CORRIGENDUM - ON THE NIELSEN-THOMSEN SEQUENCE

LAURENT CANTIER

ABSTRACT. The construction carried out in Section 4.C of the manuscript, exhibiting two *-homomorphisms from $C(\mathbb{T})$ into an AI-algebra that are only distinguished by the Hausdorffized unitary Cuntz semigroup, contained a flaw that we expose here.

The corrigendum is focused on establishing sufficient conditions for the (unitary) Cuntz semigroup to 'recover' the Hausdorffized unitary Cuntz semigroup. As a consequence, we provide a proof that the *-homomorphisms considered are *not* distinguished by the Hausdorffized unitary Cuntz semigroup and shed light on necessary conditions for building such an example.

1. The issue

The proof of [Theorem 4.10] is based on a 'local version' of the de la Harpe-Skandalis determinant, at a specific ideal I of $C(\mathbb{T})$. However, a confusion involving the elements $u_n, v_n \in C[0,1] \otimes M_{2^n}$, instead of their respective images $\phi_{n\infty}(u_n), \phi_{n\infty}(v_n) \in A$, induces a flaw in the third computation.

Indeed, we must compute the de la Harpe-Skandalis determinant of $(u'_n)_I := \varphi_{\phi_{n\infty}(u_n)}(\mathrm{id}_{I^\sim})$ and $(v'_n)_I := \varphi_{\phi_{n\infty}(v_n)}(\mathrm{id}_{I^\sim})$. (Instead of $(u_n)_I := \varphi_{u_n}(\mathrm{id}_{I^\sim})$ and $(v_n)_I := \varphi_{v_n}(\mathrm{id}_{I^\sim})$.) A standard normalisation of the universal trace yields

$$\overline{\Delta}((u'_n)_I) = [\frac{1}{2^n}(C+4f)]_{\mathrm{H}(A)} \quad \text{and} \quad \overline{\Delta}((v'_n)_I) = [\frac{1}{2^n}(C+4g)]_{\mathrm{H}(A)}.$$

We conclude that u and v have the same 'local' de la Harpe-Skandalis determinant at I, disproving the argument presented in the manuscript, that $\mathfrak{d}^*_{\mathrm{Cu}}(\mathrm{Cu}_{\overline{\mathrm{K}}_1}(\varphi_u), \mathrm{Cu}_{\overline{\mathrm{K}}_1}(\varphi_v)) > 0$. We note that it does not yet prove that the *-homomorphisms agree on the Hausdorffized unitary Cuntz semigroup. This is the main focus of this corrigendum.

2. The obstruction theorem

While aiming for an adjustment of the latter construction, we have surprisingly found rather strong obstructions, with a twofold use. First, they offer a rigorous proof that the *-homomorphisms of our original construction *do agree* on the Hausdorffized unitary Cuntz semigroup. Also, while they enlighten the difficulty of exhibiting a construction satisfying [Theorem 4.10], they most importantly provide *necessary conditions* to successfully do so.

Definition. Let A be a C*-algebra. We say that a closed two-sided ideal $I \in \mathrm{Lat}(A)$ satisfies the *obstruction property*, if the map $\overline{\mathrm{K}}_1(I \subseteq A)$ is an order-embedding.

It is straightforward to see that if an ideal I has the obstruction property, then the maps $\mathrm{H}(I \subseteq A)$ and $\mathrm{K}_1(I \subseteq A)$ are also order-embeddings. In practice, these conditions are easier to verify. The next lemma offers a form of converse, under additional assumptions.

2010 *Mathematics Subject Classification.* 46L35, 46M15.

Key words and phrases. Hausdorffized K_1-group, de la Harpe-Skandalis determinant, Cuntz semigroup.

Lemma. *Let I be a closed two-sided ideal of a C^*-algebra A, satisfying either of the following.*

(i) *The maps $\mathrm{H}(I \subseteq A)$ and $\mathrm{K}_1(I \subseteq A)$ are order-embeddings and $I \subseteq A$ is Nielsen-Thomsen diagonalisable.*

(ii) *The map $\mathrm{H}(I \subseteq A)$ is an order-embedding and $\mathrm{K}_1(I)$ is trivial.*

Then I satisfies the obstruction property.

Proof. Assume that I satisfies (i). Since the inclusion $I \subseteq A$ is diagonalisable, there exist Nielsen-Thomsen bases of A such that the associated rotation map of $\overline{\mathrm{K}}_1(I \subseteq A)$ is null. In other words, the map $\overline{\mathrm{K}}_1(I \subseteq A)$ is entirely determined by $\mathrm{H}(I \subseteq A)$ and $\mathrm{K}_1(I \subseteq A)$, from which the obstruction property follows.

Assume that I satisfies (ii). Then both the rotation map of $\overline{\mathrm{K}}_1(I \subseteq A)$ -in any pair of Nielsen-Thomsen bases-, and $\mathrm{K}_1(I \subseteq A)$ are automatically null, from which we obtain (i) and conclude the obstruction property. □

In the subsequent, we may abusively say that two *-homomorphisms $\phi, \psi \colon C(\mathbb{T}) \to A$ *agree on the extended de la Harpe-Skandalis determinant*, whenever there exists a Nielsen-Thomsen basis $\mathcal{C}$ of A such that $\overline{\Delta}_{\mathcal{C}}(\phi(\mathrm{id}_{\mathbb{T}})) = \overline{\Delta}_{\mathcal{C}}(\psi(\mathrm{id}_{\mathbb{T}}))$.

In that case, observe that $\overline{\Delta}_{\mathcal{D}}(\phi(\mathrm{id}_{\mathbb{T}})) = \overline{\Delta}_{\mathcal{D}}(\psi(\mathrm{id}_{\mathbb{T}}))$, for any Nielsen-Thomsen basis $\mathcal{D}$ of A. See [Lemma 3.18] and see [§3.11] for a reminder on the extended de la Harpe-Skandalis determinant.

Theorem (The Obstruction Theorem)**.** *Let A be a C^*-algebra and let $\phi \colon C(\mathbb{T}) \to A$ be a *-homomorphism. Assume that for any $I \in \mathrm{Lat}(C(\mathbb{T}))$, the ideal $I_\phi := \overline{A\phi(I)A} \in \mathrm{Lat}(A)$ satisfies the obstruction property.*

*Then for any *-homomorphism $\psi \colon C(\mathbb{T}) \to A$, the following are equivalent.*

(i) *ϕ and ψ agree on the Hausdorffized unitary Cuntz semigroup.*

(ii) *ϕ and ψ agree on the unitary Cuntz semigroup and the extended de la Harpe-Skandalis determinant.*

Proof. (i) naturally implies (ii). Conversely, assume that $\mathrm{Cu}_{\mathrm{K}_1}(\phi) = \mathrm{Cu}_{\mathrm{K}_1}(\psi)$ and that $\overline{\Delta}_{\mathcal{C}}(\phi(\mathrm{id}_{\mathbb{T}})) = \overline{\Delta}_{\mathcal{C}}(\psi(\mathrm{id}_{\mathbb{T}}))$, for some -and hence, any- Nielsen-Thomsen basis $\mathcal{C}$ of A. In particular ϕ and ψ agree on the Cuntz semigroup and the K_1-group, and the rotation maps of $\overline{\mathrm{K}}_1(\phi)$ and $\overline{\mathrm{K}}_1(\psi)$ with respect to $(\mathcal{C}_0, \mathcal{C})$, where $\mathcal{C}_0$ is the canonical Nielsen-Thomsen basis of $C(\mathbb{T})$ defined in [§3.11], are equal. Moreover, ϕ and ψ agree on H, by [Theorem 3.24]. As a consequence, ϕ and ψ also agree on $\overline{\mathrm{K}}_1$.

Now, recall that $\mathrm{Cu}_{\overline{\mathrm{K}}_1}(\phi)$ is given by the data $(\mathrm{Cu}(\phi), \{\overline{\mathrm{K}}_1(\phi_| \colon I \to I_\phi)\})$. Similarly for $\mathrm{Cu}_{\overline{\mathrm{K}}_1}(\psi)$. Let $I \in \mathrm{Lat}(C(\mathbb{T}))$. We know that $\mathrm{Cu}(\phi) = \mathrm{Cu}(\psi)$ and hence $I_\phi = I_\psi$, that we now write J_I. The obstruction property of J_I gives us that $\overline{\mathrm{K}}_1(\psi_| \colon I \to J_I) = \overline{\mathrm{K}}_1(\phi_| \colon I \to J_I)$ if and only if $\overline{\mathrm{K}}_1(I \xrightarrow{\phi_|} J_I \subseteq A) = \overline{\mathrm{K}}_1(I \xrightarrow{\psi_|} J_I \subseteq A)$. But we know that $\overline{\mathrm{K}}_1(I \xrightarrow{\phi_|} J_I \subseteq A) = \overline{\mathrm{K}}_1(I \subseteq C(\mathbb{T}) \xrightarrow{\phi} A)$. Similarly for ψ. We also have proven that $\overline{\mathrm{K}}_1(C(\mathbb{T}) \xrightarrow{\phi} A) = \overline{\mathrm{K}}_1(C(\mathbb{T}) \xrightarrow{\psi} A)$. As a consequence, we obtain that $\overline{\mathrm{K}}_1(\psi_| \colon I \to J_I) = \overline{\mathrm{K}}_1(\phi_| \colon I \to J_I)$, for any $I \in \mathrm{Lat}(C(\mathbb{T}))$, which ends the proof. □

Corollary. *Let A be a C^*-algebra and let $\phi \colon C(\mathbb{T}) \to A$ be a *-homomorphism. Assume that for any $I \in \mathrm{Lat}(C(\mathbb{T}))$, the ideal $I_\phi := \overline{A\phi(I)A} \in \mathrm{Lat}(A)$ belongs to $\{0, A\}$.*

*Then for any *-homomorphism $\psi \colon C(\mathbb{T}) \to A$, the following are equivalent.*

(i) *ϕ and ψ agree on the Hausdorffized unitary Cuntz semigroup.*

(ii) *ϕ and ψ agree on the unitary Cuntz semigroup and the extended de la Harpe-Skandalis determinant.*

(iii) *ϕ and ψ agree on the Cuntz semigroup, the* K_1*-group and the extended de la Harpe-Skandalis determinant.*

Proof. It is straightforward that 0 and A both satisfy the obstruction property. Henceforth (i) is equivalent to (ii), by the above theorem. Also, either of them naturally imply (iii), since the unitary Cuntz semigroup is a refinement of the Cuntz semigroup and the K_1-group.

Let us prove (iii) implies (ii). Recall that $\mathrm{Cu}_{\mathrm{K}_1}(\phi)$ is given by the data $(\mathrm{Cu}(\phi), \{\mathrm{K}_1(\phi_| \colon I \to I_\phi)\})$. Similarly for $\mathrm{Cu}_{\mathrm{K}_1}(\psi)$. Let $I \in \mathrm{Lat}(C(\mathbb{T}))$. Arguing as before, we know that $I_\phi = I_\psi$, that we now write J_I. By hypothesis, $J_I \in \{0, A\}$. If $J_I = 0$, then $\mathrm{K}_1(\phi_| \colon I \to J_I) = \mathrm{K}_1(\psi_| \colon I \to J_I)$ is the zero morphism. Otherwise $J_I = A$ and in that case, $\mathrm{K}_1(\phi_| \colon I \to A) = \mathrm{K}_1(\psi_| \colon I \to A)$ if and only if $\mathrm{K}_1(I \subseteq C(\mathbb{T}) \xrightarrow{\phi_|} A) = \mathrm{K}_1(I \subseteq C(\mathbb{T}) \xrightarrow{\psi_|} A)$, which is true, since ϕ and ψ agree on the K_1-group.

As a consequence, we obtain that $\mathrm{K}_1(\psi_| \colon I \to J_I) = \mathrm{K}_1(\phi_| \colon I \to J_I)$, for any $I \in \mathrm{Lat}(C(\mathbb{T}))$, which ends the proof. □

Closing Remark. All *-homomorphisms $\phi, \psi \colon C(\mathbb{T}) \to A$ considered in [Section 4] have common 'fixed eigenvalues', which are dense on $\mathbb{T}$. This technique was used in order to easily ensure an agreement of ϕ and ψ at the level of the Cuntz semigroup. Unfortunately, this property automatically yields that $I_\phi = I_\psi = A$, for any non-zero ideal $I \in \mathrm{Lat}(C(\mathbb{T}))$. See proofs of [Section 4]. As a consequence, all the pairs of *-homomorphisms of [Section 4] satisfy the condition of the latter corollary. In particular, this provides a proof that the pair of *-homomorphisms constructed in [Section 4.C], *do agree* on the Hausdorffized unitary Cuntz semigroup, since they agree on the unitary Cuntz semigroup and the extended de la Harpe-Skandalis determinant.

Nevertheless, the results obtained in [Section 4.A] show that the Hausdorffized unitary Cuntz semigroup can distinguish A$\mathbb{T}$-algebras that the unitary Cuntz semigroup and the $\overline{\mathrm{K}}_1$-group cannot. This suggests that there should exist *-homomorphisms $\phi, \psi \colon C(\mathbb{T}) \to A$ only distinguished by the Hausdorffized unitary Cuntz semigroup, where A is either an AI or and A$\mathbb{T}$-algebra.

(**AI statement**). A wise man once said to me: 'There is no school like old school'. Quite frankly, I cannot say I am thrilled by the new school.

(All mathematical ideas, writings and proofs are solely from the author.)

Email address: lncantier@gmail.com
URL: www.laurentcantier.fr