A new invariant for C^*-algebras

Tesis Doctoral

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A new invariant for C^*-algebras

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A mes grands-pères,
Marcel et Jean,
Deux esprits remarquablement brillants.

Certifico que aquesta memòria ha estat realitzada per Laurent Cantier sota la direcció del Dr. Ramon Antoine Riolobos i Dr. Francesc Perera Domènech

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Introduction

At the beginning of the last century, theoretical physics was at a major turning point in its history. With the recent discovery of the atom, researchers observed phenomena that could not be explained by Newton's classical mechanics and Maxwell's laws. This mathematical framework was not only insufficient to describe the observed phenomena, but even more so, it led to inconsistencies. Indeed, Heisenberg showed that the contradictions between theory and experience were mainly due to the fact that the commutative context of classical physics was no longer sufficient to explain theoretically the observed phenomena and that a mathematical context involving matrix algebras would have to be considered. Matrix algebras have an unfortunate tendency not to be commutative. Just as time is not reversible... Thus, the famous quantum physics of the 20th century were born. However, let the reader be reassured, this thesis is not dealing with that!
In fact, Heisenberg's idea also gave birth to the study of this new mathematical framework: Operator algebras. The eponymous algebras introduced by von Neumann were considered first: a closed *-sub-algebra of $\mathcal{B}(H)$ for the weak topology of operators, where H is a separable Banach space of infinite dimension. Then, in the 40's, Gelfand and Naimark discovered that using few axioms, it was possible to abstractly characterize *-subalgebras of $\mathcal{B}(H)$ closed for the norm, called C^*-algebras. Let us note that the latter contains in particular the von Neumann algebras.
As a new object had appeared, the question of its study was quickly raised. Concrete examples considered were, besides the abovementioned von Neumann algebras, the finite dimensional C^*-algebras, that are direct sums of matrix algebras of different sizes over the complex numbers. Commutative C^*-algebras are also nicely characterized. There is categorical equivalence between the category of compact Hausdorff spaces and the category of commutative unital C^*-algebras. For this reason, the theory of C^*-algebras is sometimes referred to as the noncommutative topology.
Among the first non-trivial examples, we find Glimm's analysis of UHF algebras at the very end of the 50's (see [33]). These C^*-algebras of infinite dimension, unital and simple, are constructed from sequences of algebras of finite dimension. Moreover, Glimm provided a complete invariant for these algebras, namely, the supernatural number obtained from the sequence of algebras which defines the UHF -which turns out to be nothing but the group K_0 in disguise-. Then, in the early 70's, Bratteli considered a more general class: AF algebras (see [10]). As mentioned, they contain the UHF algebras, and they are also constructed from

sequences of finite dimension algebras. The difference is that AF algebras need not be simple. In 1976, the first big breakthrough happened. Using Brattelli diagrams and the K_0-group, Elliott was able to find a complete invariant for the class of AF algebras, using the now famous intertwining technique. More precisely, he proved that the scaled ordered group $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, \Gamma(A))$ classifies all AF algebras (see [24]).
This classification was later extended to other classes of algebras. Prominent examples are the AI and A$\mathbb{T}$ algebras. Those are built as inductive limits of interval and circle algebras, that is, algebras of the form $C([0,1]) \otimes A$ or $C(\mathbb{T}) \otimes A$, respectively, where A is any finite dimensional C^*-algebra. Further on, in 1989, Elliott classified the class of A$\mathbb{T}$ algebras of real rank zero, by means of the scaled ordered group $\mathrm{K}_* := \mathrm{K}_0 \oplus \mathrm{K}_1$. Around that time, he proposed two conjectures:
(1) The scaled ordered group $(\mathrm{K}_*(A), \mathrm{K}_*(A)_+, \Sigma(A))$ is a complete invariant for separable nuclear C^*-algebra of real rank zero and stable rank one.
(2) The Elliot invariant $\mathrm{Ell}(A) := ((\mathrm{K}_0(A), \mathrm{K}_0(A)_+, \Gamma(A)), \mathrm{K}_1(A), T(A), r_A)$, where $T(A)$ is the tracial simplex on A and r_A a pairing map between $T(A)$ and $\mathrm{K}_0(A)$, is a complete invariant for simple, separable, nuclear C^*-algebras.
From that point on, the problem of classification of C^*-algebras by a functorial complete invariant, often referred to as the Elliott classification program, was definitely launched. That is to say, the aim to find a functor $\mathcal{F}$ from the category of C^*-algebras to a suitable category, that would capture enough information on C^*-algebras such that for any two $A, B \in C^*$, if there exists $\alpha : \mathcal{F}(A) \simeq \mathcal{F}(B)$ in this suitable category, then there exists a *-isomorphism $\phi : A \simeq B$ and moreover $\mathcal{F}(\phi) = \alpha$.
The classification program has provided tremendous results so far, mostly in the simple case. At first, even though the conjecture seemed huge, many subclasses of simple separable nuclear C^*-algebras were classified by the Elliott invariant or by K_*. One of the first results was obtained using K_*, cited earlier (see e.g [25], [26]), that classifies a certain subclass of AH algebras of real rank zero (containing A$\mathbb{T}$ and AI algebra of real rank zero), and morphisms of A$\mathbb{T}$ algebras of real rank zero. We recall that an AH algebra is an inductive limit of direct sums of building blocks of the form $PM_n(C(X))P$, where X is a compact metric space and P a projection in $M_n(C(X))$. Different subclasses, such as AH_d were subsequently introduced. Further, Elliott, Gong, Pasnicu, Li, Lin worked on the original Elliott invariant, classifying a restricted subclass of simple C^*-algebras (see e.g [26],[42], [53]), which led to the classification of simple AH algebras with slow dimension growth.
The conjecture would still hold so far and actually, even more classes of C^*-algebras appeared

to fit in: Kirchberg and Phillips established a remarkable result by classifying the class of purely infinite, simple, unital, nuclear C^*-algebras, now commonly known as Kirchberg algebras, by means of K-Theory, under the assumption of the Universal Coefficient Theorem (UCT) (see e.g [56], [46]).

It was not until before the early 2000's that the first counter-examples to the conjecture came up. Nevertheless, even though Rørdam followed by Toms constructed examples that would refute the Elliott conjecture (see [66] and [77]), only a small amount of additional information to the Elliott invariant such as the real rank or the stable rank of the C^*-algebra was needed to 'restore' the conjecture. A few years later though, Toms came out with a major counter-example that would refute the original conjecture (see [78]). We will speak about this construction in a short while. Let us point out that, after the work of many hands, the classification program is now complete and has provided a successful classification of simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT by the original Elliott invariant. (See, among many others, [36], [27], and [76].)

On the other hand, in the non simple case, even in the real rank zero case, results were far from satisfactory. For that matter, something more was needed. Out of this came a list of invariants, in increasing degree of complexity, aiming to 'merge' the original Elliott invariant together with the total K-Theory, an augmented version of K_*, to get a new invariant containing even more information about these algebras, especially in the non-simple case (see e.g [42], [35]). This year, it has finally been proved that the most complete invariant, termed Inv, is a complete invariant for AH algebras of no dimension growth with the ideal property (see [34], [35]).

In the meantime, another approach using the so-called Cuntz semigroup, was also considered. This object was first introduced by Cuntz in [22] and is constructed in a similar way as the Murray-von Neumann semigroup (that eventually yields to K_0 applying the Grothendieck construction), but considering positive elements instead of projections only. At first, this semigroup did not get much attention regarding the classification program. Together with the fact that its computation was rather complicated and that the original Cuntz semigroup did not preserve inductive limits, it did not seem to be a promising candidate.

However, in 2008, Toms provided a construction of two simple separable nuclear AH algebras that agree on the Elliott invariant, but fail to be isomorphic (see [78]). The tool used to distinguish the C^*-algebras was the Cuntz semigroup. As mentioned above, this was a major counter-example to the Elliott conjecture, because it was the first time that the conjecture could not be repaired by slightly modifying the Elliott invariant. In fact, with this example, Toms showed that the invariant would need to be extended not only with more K-Theoretical data, or noncommutative dimensional assumptions such as the real or stable ranks, but at least it should also include the Cuntz semigroup.

More or less at the same time, a completed version of the Cuntz semigroup that preserved inductive limits was constructed (see [21], [4]). Also, some computations were done (see [62], [75], [3]) and it was observed that the Cuntz semigroup entirely captures the lattice of ideals of the algebra (see [4], [20]). It was proved that, for unital, simple, separable, nuclear and $\mathcal{Z}$-stable C^*-algebras, one could functorially recover the original Elliott invariant using the Cuntz semigroup of any such algebra tensored with $C(\mathbb{T})$ (see [2]). Therefore, for the largest agreeable class that could be classified by the Elliott invariant, the latter and the Cuntz semigroup contain the same information.

Consequently, in the recent years, this semigroup has gained interest and has been extensively studied as it seems a promising tool for classification of non-simple C^*-algebras. Indeed, classification results of non-simple C^*-algebras by means of the Cuntz semigroup quickly appeared. This work, mainly done by Robert and Santiago (see [63], [70], [19]) gives a complete classification of non-simple classes of C^*-algebras, such as NCCW 1 complexes with trivial K_1 or AI algebras and more as they both classify homomorphisms by means of Cu or an extended version, written $\mathrm{Cu}^\sim$; see [18] [64], [63], [70].

The main drawback of this approach, as implicitly stated earlier, is that the Cuntz semigroup alone does not capture the K_1 information of the C^*-algebras. Thus, it would seem appropriate to create a unifying invariant that would 'merge' the information of the Cuntz semigroup and the information of the K-Theory. This is what this thesis is aiming for.

Structure of the thesis

In the first chapter, we introduce as concisely as possible notions about C^*-algebras and K-Theory, Category Theory, and the Cuntz semigroup.

In the second chapter, we introduce our invariant: the Cu_1-semigroup. We focus on stably finite algebras, and more concretely on the stable rank one case. On the one hand, this class is pleasantly large and includes prominent examples, such as the ones obtained by Toms alluded to above. Further, if A is a simple, unital, stably finite that absorbs the Jiang-Su algebra $\mathcal{Z}$, then A has stable rank one (see [67, Theorem 6.7]). Among non-simple algebras, all AF and all AI algebras fall into the class of C^*-algebras of stable rank one, as well as the one-dimensional non-commutative CW-complexes and the examples studied in [19]. On the other hand, for algebras of stable rank one Cuntz subequivalence of positive elements admits a nicer description easier to work with.

We first prove that this semigroup satisfies the Cuntz axioms to then define it in a categorical context. That is, we define a functor $\mathrm{Cu}_1 : C^* \longrightarrow \mathrm{Cu}^\sim$, where $\mathrm{Cu}^\sim$ is a suitable category alike as the category Cu. In fact, Cu is a full subcategory of $\mathrm{Cu}^\sim$ and we prove that Cu_1 is a continuous functor. We finally describe the positive and compact elements of our invariant.

In the third chapter, we first prove that elements of $\mathrm{Cu}_1(A)$ can be parametrized by the ideal lattice of A. Then, we introduce the notion of ideal for an abstract $\mathrm{Cu}^\sim$-semigroup, and we show that an ideal I of a $\mathrm{Cu}^\sim$-semigroup is again a $\mathrm{Cu}^\sim$-semigroup. Also, we consider quotients by ideals and prove that the set of ideals of a $\mathrm{Cu}^\sim$-semigroup S is a complete lattice, isomorphic to the complete lattice of ideals of the underlying Cu-semigroup of positive elements S_+. Further, we prove that $\mathrm{Cu}_1(A/I) \simeq \mathrm{Cu}_1(A)/\mathrm{Cu}_1(I)$ as $\mathrm{Cu}^\sim$-semigroups.

Moreover, under mild hypotheses, the maximal elements of a $\mathrm{Cu}^\sim$-semigroup S turn out to form an abelian group, that we write S_{max}. In the case of $\mathrm{Cu}_1(A)$, this abelian group is isomorphic to $\mathrm{K}_1(A)$.

All of the above allows us to functorially capture the information contained in $\mathrm{Cu}(I), \mathrm{K}_1(I)$ for any (closed two-sided) ideal I of a C^*-algebra A (and a fortiori, of A itself). We end the chapter with some results about exact sequences in the category $\mathrm{Cu}^\sim$: we first define this notion and then prove the functor Cu_1 preserves short exact sequences of ideals. Also, we link $\mathrm{Cu}, \mathrm{K}_1$ and Cu_1 in a short split-exact sequence of the form $0 \longrightarrow \mathrm{Cu} \longrightarrow \mathrm{Cu}_1 \longrightarrow \mathrm{K}_1 \longrightarrow 0$.

In the fourth chapter, we analyse situations in which $\mathrm{Cu}_1(A)$ is completely determined by $\mathrm{Cu}(A)$ and $\mathrm{K}_1(A)$, such as the simple case. We also compute $\mathrm{Cu}_1(A)$ for some non-simple C^*-algebras. In the process, we recall some well-known classes of C^*-algebras and their properties, such as AF, A$\mathbb{T}$, AI or NCCW 1-complexes.

In the fifth chapter, we introduce the notion of recovering functors. Indeed, we put a categorical context on how the information captured by some invariant can be retrieved by another invariant. A fortiori, how classification results can be transferred from one invariant to the other. We then apply the above to recover K_* from Cu_1 and reinterpret some already known classification results by means of Cu_1.

The sixth chapter is of a more technical nature. We focus our interest on Cu-semigroups of the form $\mathrm{Lsc}(X, \overline{\mathbb{N}})$, where X is a compact metric space of covering dimension 1. In particular, we consider useful features such as a countable basis, a metric and a semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(S, T)$ for $S, T \in \mathrm{Cu}$ of the form cited above. We finally use these tools to prove an approximate intertwining theorem adapted to specific inductive limits in the category Cu.

In the seventh chapter, we build an example of two C^*-algebras distinguished by Cu_1. That is, we build two NCCW 1 algebras A, B that are separable, unital, of stable rank one such that $\mathrm{Cu}(A) \simeq \mathrm{Cu}(B)$ (which implies $\mathrm{K}_0(A) \simeq \mathrm{K}_0(B)$) and $\mathrm{K}_1(A) \simeq \mathrm{K}_1(B)$, but $\mathrm{Cu}_1(A) \not\simeq \mathrm{Cu}_1(B)$ and hence $A \not\simeq B$. This confirms that our new invariant is bringing additional information regarding the classification of C^*-algebras, which is very promising for future classification results in the non-simple, non trivial K_1 case.

The eighth chapter could be seen as an opening chapter towards a future classification by means of Cu_1. We mention that this work has been done while visiting Professor L. Robert at the University of Louisiana, Lafayette. We first classify unitary elements of finite dimensional C^*-algebras by means of Cu to then (partially) extend this result to any AF-algebra. Then, by constructing two examples in $C[0, 1] \otimes M_{2^\infty}$ and $\mathcal{Z}$, we show that Cu is no longer sufficient to classify unitary elements and indeed some K_1 information needs to be added to pursue the classification of unitary elements. An opening line of research is to keep on investigating on this classification, using the Cu_1-semigroup instead of the Cu-semigroup and hopefully being able to classify unitary elements of more C^*-algebras such as NCCW 1 algebras with K_1-obstructions.

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Laurent

Contents

Chapter 1

Preliminaries

1.1 Introduction to C^*-algebras

1.1.1. Let us here give some general background on C^*-algebras to introduce our work. It will be far from complete, as we cannot restate everything, but we will be as concise as possible. For a more detailed background the reader is refered to, for example, [48], [23], [9].

Definition 1.1.2. A C^*-algebra A is an algebra over $\mathbb{C}$ with an involution $^* : A \longrightarrow A$, and equipped with an algebra norm $\| \, \|$, such that A is a Banach space and such that A satisfies the *C^*-property*: $\|aa^*\| = \|a\|^2$ for any $a \in A$. An immediate consequence of this is the following: for any $a \in A$, we have $\|a\| = \|a^*\|$.
A C^*-algebra A is called unital, if it has a multiplicative unit 1_A. As not all C^*-algebras are unital, we will see later that we can consider an 'approximation of a unit'and also a process to unitize A.
A *-homomorphism $\phi : A \longrightarrow B$ between two C^*-algebras A and B is a linear multiplicative map that is compatible with the involution. If A and B are unital and moreover $\phi(1_A) = 1_B$, we then say that ϕ is a unital *-homomorphism.
We say A is separable if it contains a countable dense subset.

1.1.3. (C^*-subalgebras - Ideals)
Let A be a C^*-algebra. A subalgebra $B \subseteq A$ closed under involution and norm is called a C^*-subalgebra of A.
An ideal I of A is always a closed two-sided ideal (unless explicitly mentioned). We can also naturally define a quotient A/I equipped with a quotient norm given by $\|x + I\| := \inf_{z \in I} \|x + z\|$. Both I and A/I are C^*-algebras.

Let us now make precise the structure of ideals of a C^*-algebra A. We write $\mathrm{Lat}(A) :=$ {closed two-sided ideals of A}. As for Rings, $(\mathrm{Lat}(A), \subseteq)$ is a complete lattice where, for any two $I, J \in \mathrm{Lat}(A)$, we define $I \wedge J := I \cap J$ and $I \vee J := I + J$.
Any *-homomorphism ϕ is continuous and of norm 1. Moreover, ϕ is injective if and only if it is isometric. Also, $\ker \phi$ is an ideal of A. Finally, if A is unital and ϕ is surjective, then ϕ is unital. A fortiori, B has a unit $1_B := \phi(1_A)$.

1.1.4. (Adjoining a unit)
Let A be an algebra. The (forced) unitization of A is the algebra $A^\sim := (A \times \mathbb{C}, +, .)$ where $(x, \lambda).(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$ with unit $1_{A^\sim} := (0, 1)$. Note that A sits as an ideal of $A^\sim$ through the canonical embedding $A \hookrightarrow A^\sim$ that sends $a \longmapsto (a, 0)$. Besides, we have $A^\sim/A \simeq \mathbb{C}$.
If A is already unital, then $A^\sim \simeq A \oplus \mathbb{C}$, where $A \oplus \mathbb{C}$ is a unital C^*-algebra, with component-wise operations and $1 := (1_A, 1_\mathbb{C})$. If A is not unital, $A^\sim$ is the smallest unital C^*-algebra that contains A as an ideal.
Finally, for any C^* morphism $\phi : A \longrightarrow B$, there exists a (unique) canonical unital C^* morphism, that we write $\phi^\sim$ such that the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A^\sim & \xrightarrow[\varphi^\sim]{} & B^\sim \end{array}$$

Definition 1.1.5. Let A be a unital C^*-algebra and let $a \in A$. We write $\mathcal{G}l(A)$ the set of invertible elements of A. The spectrum of a (with respect to A) is defined as $\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1_A$ is invertible in $A\}$. If A is not unital, we define $\mathrm{sp}(a) = \mathrm{sp}_{A^\sim}(a)$.
An element x of A is called:
(i) *normal* if $xx^* = x^*x$,
(ii) *self-adjoint* if $x^* = x$, that is, x is normal and $\mathrm{sp}\, x \subseteq \mathbb{R}$. We write A_{sa} to denote the set of self-adjoint elements.
(iii) *positive* if it is normal and $\mathrm{sp}\, x \subseteq \mathbb{R}_+$. We write A_+ to denote the set of positive elements.
(iv) *a projection* if $x = x^* = x^2$. We write $\mathcal{P}(A)$ to denote the set of projections.
(v) *a partial isometry* if $x^*x = p$ for some $p \in \mathcal{P}(A)$. If $x^*x = 1_A$, then x is an *isometry*.
(vi) *a unitary* if $xx^* = x^*x = 1_A$. We write $\mathcal{U}(A)$ to denote the set of unitary elements.
Note that all of those above are normal, except for partial isometries. Note that $\mathcal{P}(A) \subseteq A_+ \subseteq A_{sa}$ and that $\mathcal{U}(A) \subseteq \mathcal{G}l(A)$. Finally, we have $A_+ = \{x^*x, x \in A\}$. Also, for two elements a, b in

A, we say that $a \leq b$ in A if $(b - a) \in A_+$.
We will see that some of the above elements (and more particularly their equivalence classes under suitable relations) play a key role as far as invariants (for instance K-theory, Cuntz semigroup) and classification of C^*-algebras are concerned.

1.1.6. (Approximate units - Hereditary subalgebras - Full and strictly positive elements)
As said before, not every C^*-algebra A has a unit and even if we have seen a way to unitize it, sometimes we have to work in a non-unital context. Then the notion of approximate units comes into place. An approximate unit for a C^*-algebra A is an (upward-directed) net of positive elements in the closed unit ball of A, $(e_\lambda)_{\lambda\in\Lambda}$, such that for any $x \in A$, $a = \lim\limits_{\lambda} ae_\lambda$. Equivalently, $a = \lim\limits_{\lambda} e_\lambda a$.
Every C^*-algebra has an approximate unit, consisting of $\{a \in A_+, \|a\| < 1\}$. We say that A is σ-unital if it admits a countable approximate unit. In fact, if A is separable, it admits a countable approximate unit.
A C^* subalgebra B of A is called *hereditary* if for any $a \leq b$, with $a \in A$ and $b \in B$, then $a \in B$. For any subset $S \subseteq A$, we call $\mathrm{her}(S)$ the smallest hereditary subalgebra of A containing S. We write $\mathrm{Her}(A)$ the set of all hereditary subalgebras of A. Observe that any ideal is an hereditary subalgebra, that is, $\mathrm{Lat}(A) \subseteq \mathrm{Her}(A)$.
For any $a \in A_+$, we have $\mathrm{her}\, a = \overline{aAa}$. Observe that $I_a := \overline{AaA}$, the ideal generated by a, contains $\mathrm{her}\, a$. A *corner* of A is any hereditary subalgebra of A of the form $\overline{pAp} = pAp$, where p is a projection of A. We say that a hereditary subalgebra is *full* if it is not contained in any proper closed two-sided ideal of A. Thus, $\mathrm{her}\, a$ is a full hereditary subalgebra of A if and only if $I_a = A$.
Finally, an element $a \in A_+$ is called *full* if $I_a = A$ and *strictly positive* if $\mathrm{her}\, a = A$. The latter implies the former, and A admits a strictly positive element if and only if A is σ-unital. Furthermore, whenever A is separable (resp σ-unital) then any hereditary subalgebra, and a fortiori any ideal, is separable (resp σ-unital). Thus, in the separable case any $B \in \mathrm{Her}(A)$ is of the form $\mathrm{her}\, a$, for some $a \in A_+$, and any $I \in \mathrm{Lat}(A)$ is of the form I_a, for some $a \in A_+$.

1.1.7. (Murray-von Neumann equivalence)
Let A be a C^*-algebra. Let p, q be projections of A. We say that p is Murray-von Neumann equivalent to q, and we write $p \sim_{MvN} q$, if there exists a partial isometry v of A, such that $p = v^*v$ and $q = vv^*$. In this case, p is called the support projection v and q the range projection of v. Note that subequivalence is also considered, and we write $p \lesssim_{MvN} q$ whenever there exists a partial isometry $v \in A$ such that $p = v^*v$ and $vv^* \leq q$.

Define $\mathcal{P}_n(A) := \mathcal{P}(M_n(A))$ for every $n \in \mathbb{N}$ and $\mathcal{P}_\infty(A) := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(A)$ identifying $p \in M_n(A)$ with $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. Let us define a sum on $\mathcal{P}_\infty(A)$ as follows: for $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we define $p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, an element of $\mathcal{P}_{n+m}(A)$. We can now extend the Murray-von Neumann (sub)equivalence to $\mathcal{P}_\infty(A)$ as follows: if $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say that p is Murray-von Neumann (sub)equivalent to q, and we still write $p \sim_{MvN} q$ (respectively $p \precsim_{MvN} q$), if there exists a matrix of size $l \geq n, m$, such that $p \oplus 0_{n'} \sim_{MvN} q \oplus 0_{m'}$ (respectively $p \oplus 0_{n'} \precsim_{MvN} q \oplus 0_{m'}$), in $\mathcal{P}_l(A)$.

Definition 1.1.8. With all the tools defined above, one can define the Murray-von Neumann semigroup of a C^*-algebra A as follows: $V(A) := \mathcal{P}_\infty(A)/\sim_{MvN}$ where $[p] + [q] := [p \oplus q]$ and $[p] \leq [q]$ if $p \precsim_{MvN} q$. Note that $V(A)$ has $[0_A]$ as a neutral element and that any element $[p] \geq [0]$ for any $p \in \mathcal{P}_\infty(A)$.
We hence get a positively ordered monoid. Besides, the partial order is algebraic, since $[p] \leq [q]$ if and only if there exists $[r]$ such that $[p] + [r] = [q]$. In fact, if $p = vv^*$ and $v^*v \leq q$, then $r := q - v^*v$.

1.1.9. (Unitary elements)
Let A be a (unital) C^*-algebra. Let u, v be unitary elements of A. We say that u is homotopic to v, and we write $u \sim_h v$, if there exists a continuous map $f : [0,1] \longrightarrow \mathcal{U}(A)$, such that $f(0) = u$ and $f(1) = v$. Note that another equivalence is also considered: u is approximately unitarily equivalent to v whenever there exists a sequence of unitary elements of A $(w_n)_n$ such that $u = \lim_{n\in\mathbb{N}} w_n^* v w_n$. For now, we will only use and give details about the homotopy equivalence.
We define the connected component of the unit in $\mathcal{U}(A)$ as $\mathcal{U}_0(A) := \{u \in \mathcal{U}(A) \mid u \sim_h 1_A\}$. This is a normal subgroup of $\mathcal{U}(A)$. Let us mention some of the relevant properties:
(i) $\mathcal{U}^0(A) \simeq \langle \{e^{ih}\}_{h\in A_{sa}} \rangle$
(ii) If $\|u - v\| < 2$, then $u \sim_h v$. Hence if a unitary is such that $\mathrm{sp}(u) \neq \mathbb{T}$ then $u \in \mathcal{U}_0(A)$.
(iii) [Whitehead Lemma] $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$. In particular $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1_A & 0 \\ 0 & 1_A \end{pmatrix}$.
Define, for every $n \in \mathbb{N}$, $\mathcal{U}_n(A) := \mathcal{U}(M_n(A))$ and $\mathcal{U}_\infty(A) := \bigcup_{n\in\mathbb{N}} \mathcal{U}_n(A)$. Let us define a sum on $\mathcal{U}_\infty(A)$ as follows: for $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we define $u \oplus v := \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, an element of $\mathcal{U}_{n+m}(A)$. We can now extend the homotopy equivalence to $\mathcal{U}_\infty(A)$ as follows: let $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say that u is homotopic to v, and we still write $u \sim_h v$ if there exists a matrix of size $l \geq n, m$, such that $u \oplus 1_{n'} \sim_h q \oplus 1_{m'}$ in $\mathcal{U}_l(A)$.

1.1.10. (K-Theory)
We will briefly define what are the famous K-theoretical invariants for C^*-algebras: K_0 and K_1. Let us consider the unital case here.

1.1.11. (The Grothendieck Construction)
Let S be a monoid. We define an equivalence relation on $S \times S$ as follows: $(s_1, t_1) \sim (s_2, t_2)$ if there exists e in S such that $s_1 + t_2 + e = s_2 + t_1 + e$. Let us denote $[(s,t)]$ the class of the pair $(s,t) \in S \times S$. We can naturally define a component-wise addition on this quotient. We obtain an abelian group whose neutral element is $[0,0] = [(s,s)]$ and for any $(s,t) \in S \times S, [(s,t)] + [(t,s)] = [(0,0)]$. This process is called the Grothendieck construction and $\mathrm{Gr}(S) := ((S \times S)/\!\sim, +)$ is called the *Grothendieck group associated to S*.
Note that if S has cancellation, that is, $x + y = z + y$ implies $x = y$ for any $x, y, z \in S$, the construction can be simplified as follows: $(s_1, t_1) \sim (s_2, t_2)$ if $s_1 + t_2 = s_2 + t_1$.

Definition 1.1.12. Let A be a unital C^*-algebra.
(i) We define $\mathrm{K}_0(A) := \mathrm{Gr}(V(A))$. In fact, K_0 is functor from the category of unital C^*-algebras, that we write C^*, to the category of abelian groups, that we write AbGp.
(ii) We define $\mathrm{K}_1(A) := \mathcal{U}_\infty(A)/\!\sim_h$ and we define an addition on $\mathrm{K}_1(A)$ as follows: for any u, v in $\mathcal{U}_\infty(A)$, $[u] + [v] := [u \oplus v]$. $(\mathrm{K}_1(A), +)$ becomes an abelian group whose neutral element is $[1_A]$ and for any $u \in \mathcal{U}_\infty(A)$, $[u] + [u^*] = [1_A]$. In fact, K_1 is a functor from C^* to the category of abelian groups AbGp. See Section 1.2.

Theorem 1.1.13. *[6-term exact sequence]*
Let $0 \longrightarrow I \overset{\phi}{\longrightarrow} A \overset{\psi}{\longrightarrow} B \longrightarrow 0$ *be a short exact sequence in* C^*. *Then the following* 6-term *sequence is exact:*

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \xrightarrow{\mathrm{K}_0(\phi)} & \mathrm{K}_0(A) & \xrightarrow{\mathrm{K}_0(\psi)} & \mathrm{K}_0(B) \\ {\scriptstyle \delta_1}\big\uparrow & & & & \big\downarrow{\scriptstyle \delta_0} \\ \mathrm{K}_1(B) & \xleftarrow[\mathrm{K}_1(\psi)]{} & \mathrm{K}_1(A) & \xleftarrow[\mathrm{K}_1(\phi)]{} & \mathrm{K}_1(I) \end{array}$$

where δ_0 *is the so-called* exponential map *and* δ_1 *is the so-called* index map*; see [9, Theorem 9.3] for instance.*

Definition 1.1.14. Let A be a C^* algebra.
(i) If A is unital, we say that A has *stable rank one*, and we write $sr(A) = 1$, if the set of invertibles is dense in A, that is, $\overline{\mathcal{G}l(A)} = A$. This is an essential notion that has a lot of implications in many places, such as cancellation of projections in $V(A)$, or K_1-surjectivity,

among many others. Actually, we will only consider C^*-algebras of stable rank one in the rest of the thesis. See [60]. If A is non-unital, then we say that A has stable rank one if $A^\sim$ has stable rank one.
(ii) If A is unital, we say that A has *real rank zero*, and we write $RR(A) = 0$, if the set of invertible self-adjoint elements are dense in the set of self-adjoint elements, that is $\overline{A_{sa}^{-1}} = A_{sa}$. This notion ensures that the algebra has a fair number of projections. See [14]. If A is non-unital, then we say that A has real rank zero if $A^\sim$ has real rank zero.
Actually, let us give a few characterizations that are more commonly used in other parts of the thesis.

Theorem 1.1.15. *(Characterization) [14, Theorem 2.6]*
Let A be a (unital separable) C^*-algebra. Then the following are equivalent:
(i) A has real rank zero.
(ii) The set of self-adjoint elements with finite spectrum is a dense subset of A_{sa}.
(iii) For any $x \in A_{sa}$, there exist pairwise orthogonal projections $(p_i)_{i\in\mathbb{N}}$ such that $\sum_1^n \lambda_i p_i \underset{n\to\infty}{\longrightarrow} x$.
(iv) Every hereditary subalgebra has a countable approximate unit consisting of projections.
(v) Every hereditary subalgebra is generated by its projections.

Definition 1.1.16. Let A be a C^*-algebra. We say that A has the *ideal property* if any ideal of A is generated (as an ideal) by its projections. This notion is a generalization of real rank zero and any simple C^*-algebra also has the ideal property. This has been widely investigated, among others, by C. Pasnicu and we refer the reader to [53] and [52] for some properties, uses and characterizations of the ideal property.

Proposition 1.1.17. *[61, Theorem 2.10], [50, Proposition 4]*
Let A be a C^*-algebra and let $I \in \mathrm{Lat}(A)$.
If A has stable rank one, then $\mathcal{U}(A)/\mathcal{U}_0(A) \simeq \mathrm{K}_1(A)$. We also have that $\mathrm{K}_0(I) \longrightarrow \mathrm{K}_0(A)$ is injective and $\mathrm{K}_1(A) \longrightarrow \mathrm{K}_1(A/I)$ is surjective.
If A has real rank zero, then $\mathrm{K}_1(I) \longrightarrow \mathrm{K}_1(A)$ is injective and $\mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(A/I)$ is surjective.

Theorem 1.1.18 (Künneth Formula)**.** *([9, Theorem 23.1.3]) Let A, B be C^*-algebra such that $\mathrm{K}_0(A) \oplus \mathrm{K}_1(B)$ is torsion-free. Then:* $\mathrm{K}_0(A \otimes B) \simeq \mathrm{K}_0(A) \otimes \mathrm{K}_0(B) \oplus \mathrm{K}_1(A) \otimes \mathrm{K}_1(B)$.
$\mathrm{K}_1(A \otimes B) \simeq \mathrm{K}_0(A) \otimes \mathrm{K}_1(B) \oplus \mathrm{K}_1(A) \otimes \mathrm{K}_0(B)$.

1.1.19. (Examples) We will here give some basic examples of C^*-algebras and we will in the meantime illustrate some of the notions introduced above.

(i) The algebra of bounded operators. Let H be any Hilbert space and consider $\mathcal{B}(H)$ the set of all bounded linear operators on H. Equipped with adjoint as the involution and the operator norm, $\mathcal{B}(H)$ is a C^*-algebra. One can also consider $\mathcal{K}(H)$, the norm-closed span of all operators with finite-dimensional range in $\mathcal{B}(H)$. It is also a C^*-algebra and the unique non-trivial ideal of $\mathcal{B}(H)$.
• Take $H = \mathbb{C}^n$ for any $n \in \mathbb{N}$. Then $\mathcal{B}(\mathbb{C}^n) = \mathcal{K}(\mathbb{C}^n) = M_n(\mathbb{C})$, and it is a C^*-algebra of 'finite dimension'. Note that in this case, the involution of an element M is its transpose-conjugate $M^* = \overline{M}^t$.
• Take $H = l^2(\mathbb{N})$, any separable infinite dimensional Hilbert space (up to isometry). Then we write $\mathcal{K} := \mathcal{K}(l^2(\mathbb{N}))$. This is an important case. Indeed, we have $A \otimes \mathcal{K} = \varinjlim_n (M_n(A), i_n)$, where i_n are the canonical inclusions. We call $A \otimes \mathcal{K}$ the *stabilization* of A. Furthermore $\mathrm{K}_1, \mathrm{K}_0, \mathrm{Lat}$ are stable under stabilization, that is, $\mathrm{Lat}(A \otimes \mathcal{K}) \simeq \mathrm{Lat}(A)$, respectively $\mathrm{K}_0, \mathrm{K}_1$.
In general, one can observe that for any C^*-algebra A and any $n \in \mathbb{N}$, we have $A \otimes M_n(\mathbb{C}) \simeq M_n(A)$ as follows: take $a \in A$ and $m := (m_{ij})_{1 \leq i,j \leq n} \in M_n$. Then:

$$a \otimes m = \begin{pmatrix} m_{11}a & m_{12}a & \dots & m_{1n}a \\ m_{21}a & m_{22}a & \dots & m_{2n}a \\ \vdots & \vdots & & \vdots \\ m_{n1}a & m_{n2}a & \dots & m_{nn}a \end{pmatrix}$$

(ii) The commutative setting. Let X be a locally compact Hausdorff topological space. One can define $C_0(X)$ as the set of continuous complex-valued maps over X that vanish at infinity, that is, continuous maps $f : X \longrightarrow \mathbb{C}$ such that for any $\epsilon > 0$, the set $\{x \in X, |f(x)| \geq \epsilon\}$ is compact. Equipped with the point-wise operation, the supremum norm and the involution induced by complex conjugation, that is $f^*(x) = \overline{f(x)}$, the algebra $C_0(X)$ is a C^*-algebra.
Actually, any commutative C^*-algebra is of this form and $C_0(X) \simeq C_0(Y)$ if and only if $X \simeq Y$. Moreover $C_0(X)$ is unital if and only if X is compact, and in this case $C_0(X) = C(X)$, the set of all continuous complex-valued maps over X.

(iii) One can now start to combine the existing basic examples we already have to get other C^*-algebras. For instance, any direct sums of matrices over the complex numbers, that is, $\bigoplus_i M_{n_i}(\mathbb{C})$ is a C^*-algebra. These are the finite dimensional C^*-algebras, up to *-isomorphisms. One can also obtain other examples computing direct limits (see Section 1.2) of direct sums of the examples above. This is how we will define the approximately finite dimensional C^*-

algebras, also called AF algebras (which are nothing more than direct limits of finite dimensional C^*-algebras), among other classes of C^*-algebras, such as AI, A$\mathbb{T}$ and AH algebras. Those will be defined in detail later.

(iv) [Gelfand-Naimark] Any C^*-algebra A can be embedded (isometrically) into $\mathcal{B}(H)$ for some Hilbert space H. This is most commonly known as the Gelfand-Naimark theorem. Actually, the Gelfand-Naimark-Segal construction, known as the GNS construcion, explicitly gives us a Hilbert space H_u and faithful representation $\pi : A \longrightarrow \mathcal{B}(H_u)$, that is, an injective *-homomorphism (hence an isometry) from A into $\mathcal{B}(H_u)$. This map is sometimes refered to as the *universal representation of* A. As it turns out that whenever A is separable, H_u can be chosen separable too. Also, this allows us to give another *concrete* definition (as opposed to the *abstract* one we already have) of a C^*-algebra: a *concrete* C^*-algebra is a norm-closed, *-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

1.1.20. We next remind the reader about the bidual of a C^*-algebra, connections between positive elements of $A \otimes \mathcal{K}$, countably generated right A-Hilbert modules, open/support projections and the hereditary subalgebra generated by an element of $A \otimes \mathcal{K}$. Until the end of this section, everything stated can be found in [48] and in [58].

Definition 1.1.21. Let H be an (infinite dimensional) Hilbert space. We can define various topologies on $\mathcal{B}(H)$ as follows:
(i) The *norm topology* naturally arising from the operator norm of $\mathcal{B}(H)$. A sequence of operators $(T_n)_n$ converges towards T if $\|T_n - T\| \underset{n}{\rightarrow} 0$.
(ii) The *strong operator topology*, written SOT: A sequence of operators $(T_n)_n$ converges strongly towards T if for any $h \in H$, $\|T_n(h) - T(h)\|_H \underset{n}{\rightarrow} 0$.
(iii) The *weak operator topology*, written WOT: A sequence of operators $(T_n)_n$ converges weakly towards T if for any h, h' in H, $\langle (T_n - T)(h), h' \rangle_H \underset{n}{\rightarrow} 0$.
Obviously, the norm topology is stronger than the SOT, which is stronger than the WOT.

Definition 1.1.22. Let A be a C^*-algebra and let $\pi : A \longrightarrow \mathcal{B}(H_u)$ be its universal representation. We call the weak closure of $\pi(A)$ in $\mathcal{B}(H_u)$ the enveloping von Neumann algebra of A.

Theorem 1.1.23. *(von-Neumann bicommutant theorem)*
The enveloping von-Neumann algebra $\overline{\pi(A)}^{WOT}$ of a C^* algebra A is equal to its strong closure $\overline{\pi(A)}^{SOT}$, and also equal to the bicommutant A'' of A.

Theorem 1.1.24. *(Sherman-Takeda)*
Let A be a C^* algebra. Then the algebraic bidual A^{**} of A is isomorphic to its bicommutant A'' (equivalently to its enveloping von-Neumann algebra) as Banach spaces.

Definition 1.1.25. We call a projection p of A^{**} an open projection if $p \in \overline{pA^{**}p \cap A}^{\text{strong}}$.
We define the support projection of a, the only open projection of A^{**} such that $A_{p_a} := \overline{p_a A^{**} p_a \cap A}^{\text{strong}} = \operatorname{her} a$. Whenever A is separable, one can prove that $p_a = SOT - \lim a^{1/n}$.

Proposition 1.1.26. *[58, §3] Let A be a C^*-algebra. Then we have the following set bijections: $P_{open}(A^{**}) \simeq \operatorname{Her}(A)$ and $P_{support}(A^{**}) \simeq \{\operatorname{her} a, a \in A_+\}$. Thus, if A is separable then $P_{support}(A^{**}) = P_{open}(A^{**})$.*

Definition 1.1.27. Let a and b be in A_+. We write $a \sim_s b$ if there exists $x \in A$ such that $\operatorname{her} a = \operatorname{her}(xx^*)$ and $\operatorname{her} b = \operatorname{her}(x^*x)$. We sometimes write $\operatorname{her} a \sim_s \operatorname{her} b$. Further, we write $a \lesssim_s b$ if there exists $a' \in (\operatorname{her} b)_+$ such that $a' \sim_s a$.
Let p and q be in $P_{open}(A^{**})$. We say that p is Peligrad-Zsidó equivalent to q, and we write $p \sim_{PZ} q$, if there exists α partial isometry of A^{**} such that $p = \alpha\alpha^*, q = \alpha^*\alpha, \alpha^* A_p \subseteq A$ and $A_q \alpha \subseteq A$. Further, we write $p \lesssim_{PZ} q$ if there exists a projection $p' \leq q$ such that $p \sim_{PZ} p'$.

Proposition 1.1.28. *[58, Proposition 4.3] Let a and b be in A_+. Then the following are equivalent:*
(i) $a \sim_s b$
(ii) $p_a \sim_{PZ} p_b$
In this case, for any partial isometry $\alpha \in A^{**}$ that realizes the Peligrad-Zsidó equivalence between p_a and p_b, we have an explicit isomorphism as follows:

$$\begin{aligned} \theta_{ab,\alpha} : \operatorname{her} a &\simeq \operatorname{her} b \\ d &\longmapsto \alpha^* d \alpha \end{aligned}$$

Remark 1.1.29. Note that in case of subequivalence only, the explicit morphism constructed above is well defined and is an injection from $\operatorname{her} a$ into $\operatorname{her} b$.

1.1.30. The next proposition is somehow similar to [58, 3.3 Proposition] and [55, Theorem 1.4], but for the sake of completeness we will give a proof of it in this slightly different picture.

Proposition 1.1.31. *Let p be a support projection in A^{**}. Let a in A_+ such that $p = p_a$. Let α be a partial isometry in A^{**} such that $p = \alpha\alpha^*$. We set $q := \alpha^*\alpha$ and $x := a^{1/2}\alpha$. Then $p \sim_{PZ} q$ if and only if x belongs to A. In this case, $q = p_{x^*x}$.*

Proof. The forward implication is coming from the definition of the Peligrad-Zsidó equivalence itself.
Now let us suppose that $x := a^{1/2}\alpha$ belongs to A. Let d be in aAa. Then there exists δ_d in A such that $d = a\delta_d a$. Now observe that $\alpha^* d = \alpha^* a^{1/2} a^{1/2}\delta_d a$, so it belongs to A, using the hypothesis that x is in A. We get that $\alpha^* aAa \subseteq A$, so $\overline{\alpha^* aAa} \subseteq A$, from which we deduce $\alpha^* A_p \subseteq A$.
One can see that because p is a support projection and $q = \alpha^* p\alpha$, we have that q is a support projection and moreover $\alpha^* A_p \alpha = A_q$. Using the fact that $\alpha A_q = \alpha A_q \alpha^* \alpha = A_p \alpha$ and that $(\alpha^* A_p)^* = A_p \alpha$, we deduce that $\alpha A_q \subseteq A$. We conclude that $p \sim_{PZ} q$ and by construction $q = p_{x^* x}$. □

Corollary 1.1.32. *Let p be a support projection in A^{**}. Let q be a projection such that $q \sim_{MvN} p$ in A^{**}. Then q is a support projection if and only if $q \sim_{PZ} p$.*

1.2 Introduction to Category Theory

1.2.1. In this section we briefly recall some concepts on Category Theory. Again, this is far from complete but we refer the reader to [47] for more details.

1.2.2. (Basic definitions)
A *category* $\mathcal{C}$ is a collection of *objects* that are linked by *arrows*. We require that the arrows compose in an associative way and that there exists an identity arrow, that we write id_C, for any object C in the category. We usually refer to an arrow as a *morphism*. Also, we usually denote the collection of objects of $\mathcal{C}$ by $\mathrm{Ob}(\mathcal{C})$ or simply by $\mathcal{C}$ and we denote the collection of morphisms from C_1 to C_2, where $C_1, C_2 \in \mathcal{C}$, by $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ or simply by $\mathcal{C}(C_1, C_2)$.
A first basic example is the category Set whose objects are sets and morphisms between two objects are any maps between those sets. Note that we will always be in the context where the collection of objects and the collection of morphisms between two objects are in fact objects in Set, that is, sets. These categories are referred to as *locally small* categories. We do not wish to go further on this topic, but we always suppose that a category is locally small.

A *covariant functor* F between two categories $\mathcal{C}, \mathcal{D}$ is an assignment: $F : \mathcal{C} \longrightarrow \mathcal{D}$ that sends any object $C \in \mathcal{C}$ to an object $F(C) \in \mathcal{D}$, and any morphism $f \in \mathcal{C}(C_1, C_2)$ between $C_1, C_2 \in \mathcal{C}$ to a morphism $F(f) \in \mathcal{D}(F(C_1), F(C_2))$ such that:
(i) $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for any $C \in \mathcal{C}$.

(ii) $F(g \circ f) = F(g) \circ F(f)$ for any $f \in \mathcal{C}(C_1, C_2)$ and any $g \in \mathcal{C}(C_2, C_3)$.
In fact, there exist functors that 'reverse' morphisms. That is, using notations of the above, any morphism $f \in \mathcal{C}(C_1, C_2)$ between $C_1, C_2 \in \mathcal{C}$ is sent to a morphism $F(f) \in \mathcal{D}(F(C_2), F(C_1))$. In this case, condition (ii) needs adaptation and we say that F is a *contravariant* functor.
Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two (locally small) categories $\mathcal{C}, \mathcal{D}$.
We say that F is *faithful* if for any two objects $C_1, C_2 \in \mathcal{C}$, the Set-morphism $F_{C_1,C_2} : \mathcal{C}(C_1, C_2) \longrightarrow \mathcal{D}(F(C_1), F(C_2))$ induced by F is an injective morphism.
We say that F is *full* if for any two objects $C_1, C_2 \in \mathcal{C}$, the Set-morphism $F_{C_1,C_2} : \mathcal{C}(C_1, C_2) \longrightarrow \mathcal{D}(F(C_1), F(C_2))$ induced by F is a surjective morphism.

Let $\mathcal{C}$ be a category. A *subcategory* of $\mathcal{C}$ is a category $\mathcal{D}$ whose objects are objects in $\mathcal{C}$ and whose morphisms are morphisms in $\mathcal{C}$ with the same identities and compositions of morphisms. There exists a natural functor $i : \mathcal{D} \longrightarrow \mathcal{C}$ that we call the *inclusion functor*. This is clearly a faithful functor. If moreover, the inclusion functor is full, we say that $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$.

1.2.3. (Limits/Colimits - Completeness/Cocompleteness)
Consider a (commutative) diagram $\Lambda := (C_i, f_{ij})_{i,j\in I}$ in $\mathcal{C}$. That is, a collection of objects of $\mathcal{C}$ linked by a collection morphisms of $\mathcal{C}$ such that any two different paths starting and ending at the object commute. Note that we will always be in the context where these two collections are in fact objects in Set, that is, sets. These diagrams (and their limits/colimits) are referred to as *small* diagrams/limits/colimits. We do not wish to go further on this topic, but we always suppose that a diagram/limit/colimit is small.
A notion of *limit* and dually a notion of *colimit* of a diagram can be defined in a category $\mathcal{C}$ as follows:
A *cone to* Λ is a pair $(C, f_{i\infty})_{i\in I}$, where $f_{i\infty} : C \longrightarrow C_i$ is a $\mathcal{C}$-morphism, such that for any $f_{ij} : C_i \longrightarrow C_j$ in I, we have $f_{ij} \circ f_{i\infty} = f_{j\infty}$. A *limit of* Λ is a cone to Λ, $(C, f_{i\infty})_{i\in I}$, such that for any other cone $(C', f'_{i\infty})_{i\in I}$ there exists a unique $\mathcal{C}$-morphism $u : C' \longrightarrow C$ such that $f'_{i\infty} = f_{i\infty} \circ u$ for any $i \in I$. Inverse limits, pullbacks, (infinite) direct products are examples of (small) limits.
Dually, a *cocone to* Λ is a pair $(C, f_{i\infty})_{i\in I}$, where $f_{i\infty} : C_i \longrightarrow C$ is a $\mathcal{C}$-morphism such that for any $f_{ij} : C_i \longrightarrow C_j$ in I, we have $f_{j\infty} \circ f_{ij} = f_{i\infty}$. A *colimit of* Λ is a cone to Λ, $(C, f_{i\infty})_{i\in I}$, such that for any other cone $(C', f'_{i\infty})_{i\in I}$ there exists a unique $\mathcal{C}$-morphism $u : C \longrightarrow C'$ such that $f'_{i\infty} = u \circ f_{i\infty}$ for any $i \in I$. Inductive limits (or direct limits), pushouts, (infinite) direct sums (or coproducts) are examples of (small) colimits.

Note that a limit or a colimit need not exist but when it does, it is unique up to isomorphism. Thus we speak about the limit/colimit of a diagram.
Finally, we say a category $\mathcal{C}$ is *complete* if any diagram admits a limit in $\mathcal{C}$. Dually, we say a category $\mathcal{C}$ is *cocomplete* if any diagram admits a colimit in $\mathcal{C}$.

1.2.4. (Adjunction - Reflection/Coreflection)
Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be (covariant) functors between two categories $\mathcal{C}, \mathcal{D}$. We say that F is *left-adjoint to* G, and hence, G is *right-adjoint to* F if for any $C \in \mathcal{C}$ and any $D \in \mathcal{D}$, we have $\mathcal{C}(C, G(D)) \simeq \mathcal{D}(F(C), D)$ in a natural way. We will specify what the notion of naturality means when needed.

Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. We say $\mathcal{D}$ is a *reflective subcategory* of $\mathcal{C}$ if the inclusion functor i has a left-adjoint, that we call a *reflector*. Dually, we say $\mathcal{D}$ is a *coreflective subcategory* of $\mathcal{C}$ if the inclusion functor i has a right-adjoint, that we call a *coreflector*.
Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$. Then any colimit in $\mathcal{C}$ passes through reflectors to a colimit in $\mathcal{D}$. Dually, limits pass to coreflective subcategories through coreflectors.

1.2.5. (Examples) We will here give some basic examples of categories and illustrate some of the notions introduced above.
The category of *abelian groups*, that we write AbGp, whose objects are abelian groups and morphisms are group homomorphisms. It is a full subcategory of the category of groups. It is a bicomplete category.
The category of *C^*-algebras*, that we write C^*, whose objects are C^*-algebras and morphisms are *-homomorphisms. It is a bicomplete category.
The category of *positively ordered monoids*, that we write PoM, whose objects are positively ordered monoids and morphisms are monoid morphisms that respect $\leq$. It has inductive limits, products and coproducts. Let us here explicitly construct an inducitve limit in PoM, knowing that an adapted construction could be done in AbGp or C^*:
Let $(S_i, \varphi_{ij})_{i\in I}$ be an inductive system in PoM. We define $S := \bigsqcup_{i\in I} S_i/\sim$, where $x \sim y$ for $x \in S_i$ and $y \in S_j$, if there exists $k \geq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(x)$. We equip S with $+$ and $\leq$ as follows: Let x and y be in S. We define $x + y := \varphi_{ik}(x_i) + \varphi_{jk}(y_j)$ and we say $x \leq y$ if $\varphi_{ik}(x_i) \leq \varphi_{jk}(y_j)$, where $x_i \in S_i$ and $y_j \in S_j$ are representatives of x and y respectively and $k \geq i, j$. One can check that $(S, +, \leq) \in$ PoM and that it is the inductive limit of the diagram in the category PoM.

1.3 The Cuntz semigroup of a C^*-algebra

1.3.1. We will recall some definitions and properties that will lead us to the Cuntz semigroup of a C^*-algebra. More details can be found in [4], [71], [64], [63], [21].

Definition 1.3.2. Let A be a C^*-algebra. We denote by A_+ the set of positive elements. Let a and b be in A_+. We say that a is Cuntz subequivalent to b, and we write $a \lesssim_{\mathrm{Cu}} b$, if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $a = \lim\limits_{n\in\mathbb{N}} x_n b x_n^*$. After antisymmetrizing this relation, we get an equivalence relation over A_+, called the Cuntz equivalence, denoted by $\sim_{\mathrm{Cu}}$.
Let us write $\mathrm{Cu}(A) := (A\otimes\mathcal{K})_+/\!\sim_{\mathrm{Cu}}$, that is, the Cuntz classes of positive elements of $A\otimes\mathcal{K}$. Given $a\in(A\otimes\mathcal{K})_+$, we write $[a]$ the Cuntz class of a. Now we use the isomorphism between $M_2(A\otimes\mathcal{K})\simeq A\otimes\mathcal{K}$ to define an addition as follows: let v_1 and v_2 be two isometries in the multiplier algebra of $A\otimes\mathcal{K}$, such that $v_1v_1^* + v_2v_2^* = 1_{M(A\otimes\mathcal{K})}$. Consider the *-isomorphism $\psi : M_2(A\otimes\mathcal{K}) \longrightarrow A\otimes\mathcal{K}$ given by $\psi(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}) = v_1av_1^* + v_2bv_2^*$, and we write $a\oplus b := \psi(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix})$.
The set $\mathrm{Cu}(A)$ comes with a natural order given by $[a]\leq[b]$ whenever $a\lesssim_{\mathrm{Cu}} b$, and we set $[a]+[b] := [a\oplus b]$ for any $[a],[b]$ in $\mathrm{Cu}(A)$. In this way $\mathrm{Cu}(A)$ is now a semigroup called *the Cuntz semigroup of A*.
For any *-homomorphism $\phi : A\longrightarrow B$, one can define $\mathrm{Cu}(\phi) : \mathrm{Cu}(A)\longrightarrow\mathrm{Cu}(B)$, a semigroup map, by $[a]\longmapsto[(\phi\otimes id_{\mathcal{K}})(a)]$. Hence, we get a functor from the category of C^*-algebras into a certain subcategory of PoM, called the category Cu, that we describe next.

Proposition 1.3.3. *[19, Proposition 1] Let A be a C^*-algebra of stable rank one and let $a,b\in A_+$. Then $a\lesssim_{\mathrm{Cu}} b$ if and only if there exists $x\in A$ such that $xx^* = a$ and $x^*x\in \mathrm{her}\, b$.*

Proposition 1.3.4. *[58, §6] Let A be a C^*-algebra and let $a,b\in A_+$.*
(i) If $a\sim_s b$ (resp $a\lesssim_s b$), then $a\sim_{Cu} b$ (resp $a\lesssim_{\mathrm{Cu}} b$).
(ii) If $p_a\sim_{PZ} p_b$ (resp $p_a\lesssim_{PZ} p_b$), then $p_a\sim_{Cu} p_b$ (resp $p_a\lesssim_{Cu} p_b$).
(iii) Finally, if A has stable rank one then all converse implications hold.

Remark 1.3.5. As we will exclusively be concerned with the stable rank one case, we have equivalences between $a\sim_s b$, $a\sim_{\mathrm{Cu}} b$, and $p_a\sim_{PZ} p_b$ (resp $\lesssim_*$).

Definition 1.3.6. Let $(S,\leq)$ be a positively ordered semigroup. An *auxiliary relation* on S is a binary relation $\prec$ such that:
(i) $\mathcal{R}_\prec\subseteq\mathcal{R}_\leq$.
(ii) For any $a,b,c,d\in S$ such that $a\leq b\prec c\leq d$ then $a\prec d$.
(iii) For any $a\in S$, we have $(0,a)\in\mathcal{R}_\prec$.

Definition 1.3.7. Let $(S, \leq)$ be a positively ordered semigroup. For any x, y in S, we say that x is *way-below* y and we write $x \ll y$ if, for any increasing sequence $(z_n)_{n\in\mathbb{N}}$ that has a supremum in S such that $\sup\limits_{n\in\mathbb{N}} z_n \geq y$, there exists k such that $z_k \geq x$. This is an auxiliary relation on S called the *compact-containment relation*. In particular $x \ll y$ implies $x \leq y$.
We say that S is an abstract Cu-semigroup if it satisfies the Cuntz axioms:
(O1): Every increasing sequence of elements in S has a supremum.
(O2): For any $x \in S$, there exists a $\ll$-increasing sequence $(x_n)_{n\in\mathbb{N}}$ in S such that $\sup\limits_{n\in\mathbb{N}} x_n = x$.
(O3): Addition and the compact containment relation are compatible.
(O4): Addition and suprema of increasing sequences are compatible.
A *generalized* Cu-*morphism* between two Cu-semigroups S, T is a positively ordered monoid that preserves suprema of increasing sequences. A Cu-*morphism* between two Cu-semigroups S, T is a positively ordered monoid that preserves the compact containment relation and suprema of increasing sequences.
The Cuntz category, written Cu is the subcategory of PoM whose objects are Cu-semigroups and morphims are Cu-morphisms. Actually, as shown for instance in [4, Corollary 3.2.9], the functor Cu from the category of C^*-algebras to Cu is arbitrarily continuous, generalizing the result of [21, Theorem 2] that established sequential continuity.

Definition 1.3.8. Let S be a Cu-semigroup. We say that S is *countably-based* if there exists a countable subset $B \subseteq S$ such that for any $a, a' \in S$ such that $a' \ll a$, then there exists $b \in B$ such that $a' \leq b \ll a$. The set B is often referred to as a *basis*.
An element $u \in S$ is called an *order-unit* of S if for any $x \in S$, there exists $n \in \overline{\mathbb{N}}$ such that $x \leq n.u$.

1.3.9. Let S be a countably-based Cu-semigroup. Then, S has a maximal element, or equivalently, it is singly-generated. Let us also mention that if A is a separable C^*-algebra, then $\mathrm{Cu}(A)$ is countably-based. In fact, its largest element, that we write ∞_A, can be explicitly constructed as follows: Let s_A be any strictly positive element (or full) in A. Then $\infty_A = \sup\limits_{n\in\mathbb{N}} n.[s_A]$. A fortiori, $[s_A]$ is an order-unit of $\mathrm{Cu}(A)$.

1.3.10. (Lattice of ideals in Cu)
Let S be a Cu-semigroup. An *ideal* of S is a submonoid I that is closed under suprema of increasing sequences and such that for any x, y such that $x \leq y$ and $y \in I$, then $x \in I$.
It is shown in [4, §5.1.6], that for any I, J ideals of S, $I \cap J$ is again an ideal. Therefore for any $x \in S$, the ideal generated by x, defined as the smallest ideal of S containing x, that we write

I_x, is exactly the intersection of all ideals of S containing x. An explicit computation gives us $I_x := \{y \in S \text{ such that } y \leq \infty.x\}$.
Moreover it is shown that $I + J := \{z \in S \mid z \leq x + y, x \in I, y \in J\}$ is also an ideal. Thus we write $\mathrm{Lat}(S) := \{\text{ideals of } S\}$, which is a complete lattice under the following operations: for any two $I, J \in \mathrm{Lat}(S)$, we define $I \wedge J := I \cap J$ and $I \vee J := I + J$.
Furthermore, for any C^*-algebra A, we have that $\mathrm{Cu}(I)$ is an ideal of $\mathrm{Cu}(A)$ for any $I \in \mathrm{Lat}(A)$. In fact, we have a lattice isomorphism as follows:

$$\begin{aligned} \mathrm{Lat}(A) &\overset{\simeq}{\longrightarrow} \mathrm{Lat}(\mathrm{Cu}(A)) \\ I &\longmapsto \mathrm{Cu}(I) \end{aligned}$$

Finally, whenever S is countably-based, any ideal I of S is singly-generated, for instance by its largest element, that we also write ∞_I. In particular, for any separable C^*-algebra A, any $a, b \in (A \otimes \mathcal{K})_+$, if $[a] \leq [b]$ in $\mathrm{Cu}(A)$, then $I_a \subseteq I_b$, or equivalently $I_{[a]} \subseteq I_{[b]}$. (Notice that the converse is a priori not true: $I_x = I_{k.x}$ for any $x \in \mathrm{Cu}(A)$, any $k \in \overline{\mathbb{N}}$ but in general $x \neq k.x$).

1.3.11. (Quotients in Cu)
Let S be a Cu-semigroup and $I \in \mathrm{Lat}(S)$. Let $x, y \in S$. We write $x \leq_I y$ if: there exists $z \in I$ such that $x \leq z + y$. By antisymmetrizing $\leq_I$, we obtain an equivalence relation $\sim_I$ on S. Define $S/I := S/{\sim_I}$. For $x \in S$, write $\overline{x} := [x]_{\sim_I}$ and equip S/I with the following addition and order: Let $x, y \in S$. Then $\overline{x} + \overline{y} := \overline{x + y}$ and $\overline{x} \leq \overline{y}$, if $x \leq_I y$. These are well-defined and $(S/I, +, \leq)$ is a Cu-semigroup, often referred to as the *quotient of S by I*. Moreover, the canonical quotient map $S \longrightarrow S/I$ is a surjective Cu-morphism. Finally, for any C^*-algebra A and any $I \in \mathrm{Lat}(A)$, we have $\mathrm{Cu}(A/I) \simeq \mathrm{Cu}(A)/\mathrm{Cu}(I)$; see [20, Corollary 2].

Definition 1.3.12. Let X be a topological space and S be a Cu-semigroup. Let $f : X \longrightarrow S$ be a map. We say that f is *lower-semicontinuous* if for any $s \in S$, the set $\{t \in X : s \ll f(t)\}$ is open in X. We write $\mathrm{Lsc}(X, S)$ the set of lower-semicontinuous functions from X to S.

1.3.13. In the following see Definition 6.1.1 for the definition of covering dimension.

Theorem 1.3.14. *[3, Theorem 5.15]*
Let X be a compact Hausdorff second countable space of finite covering dimension and let S be a countably-based Cu*-semigroup. Then* $\mathrm{Lsc}(X, S)$ *is also a* Cu*-semigroup.*

Theorem 1.3.15. *[3, Theorem 3.4] Let A be a separable C^*-algebra of stable rank one such that* $\mathrm{K}_1(I) = 0$ *for every ideal of A. Let X be a locally compact Hausdorff space that is second countable and of covering dimension one. Then* $\mathrm{Cu}(C_0(X) \otimes A) \simeq \mathrm{Lsc}(X, \mathrm{Cu}(A))$.

1.3.16. We end this section by defining the notion of algebraic Cu-semigroups. This allows us to link the real rank zero property of a C^*-algebra A, that ensures a lot of projections with the notion of 'density' of compact elements in $\mathrm{Cu}(A)$. We refer the reader to [4, §5.5] for details.
Let $S \in \mathrm{Cu}$. Using the Cuntz axioms, one can check that $S_c := \{x \in S \mid x \ll x\}$ is a PoM. An element $x \in S_c$ is referred to as a *compact element*. Furthermore, we know that for any Cu-morphism $f : S \longrightarrow T$ between two Cu-semigroups S, T, $f(S_c) \subseteq T_c$, so f induces a PoM-morphism $f_c : S_c \longrightarrow T_c$.
Thus, we define the following functor:

$$\begin{aligned} \nu_c : \mathrm{Cu} &\longrightarrow \mathrm{PoM} \\ S &\longmapsto S_c \\ f &\longmapsto f_c \end{aligned}$$

On the other hand for any $M \in \mathrm{PoM}$ such that $M \subseteq S$, we define the 'completion' of M in S, that we write $\gamma(M)$, as the subset of S consisting of suprema (in S) of any increasing sequence in M. One can check that $(\gamma(M), \leq) \in \mathrm{Cu}$.

Definition 1.3.17. Let $S \in \mathrm{Cu}$. We say S is an *algebraic* Cu-*semigroup* if any $s \in S$ is the supremum of an increasing sequence of compact elements. That is, an increasing sequence in S_c. We denote by Cu_{alg} the full subcategory of Cu consisting of algebraic Cu-semigroups.

Proposition 1.3.18. *[4, Proposition 5.5.4]*
(i) For any $S \in \mathrm{Cu}_{alg}$, we have $\gamma(S_c) \simeq S$ as Cu-*semigroups.*
(ii) For any $S \in \mathrm{Cu}$ and any $M \in \mathrm{PoM}$ such that $M \subseteq S$, we have $(\gamma(M), \leq) \in \mathrm{Cu}_{alg}$.

Theorem 1.3.19. *[21, Corollary 5], [4, Remark 5.5.2]*
Whenever A has real rank zero, $\mathrm{Cu}(A)$ *is an algebraic* Cu-*semigroup. Moreover, if A has stable rank one, the converse is true.*

1.4 Traces and Functionals

All of the following can be found in [30] and [71].

Definition 1.4.1. Let A be a C^*-algebra. Let $\tau : A_+ \longrightarrow \overline{\mathbb{R}}_+$. We say that τ is a *trace on A* if it is additive, homogeneous (that is, $\tau(0) = 0$ and $\tau(r.a) = r.\tau(a)$ for any $a \in A_+$, $r \in \mathbb{R}_+^*$) and satisfies the *trace property*, that is, $\tau(xx^*) = \tau(x^*x)$ for any $x \in A$.
We say that τ is *lower-semicontinuous* if it is continuous for the Scott-topology on $\mathbb{R}$, that

is, $\tau^{-1}(]t;\infty])$ are open sets of A for any $t > 0$. We denote by $T(A)$ the set of all lower-semicontinuous traces on A.

Proposition 1.4.2. *Every trace on A_+ extends uniquely to o trace on $(A \otimes \mathcal{K})_+$.*

Definition 1.4.3. A *1-quasitrace* on A is a map $\tau : A_+ \longrightarrow \overline{\mathbb{R}}_+$ that is homogeneous, additive on commutating elements, and statisfies the trace property.
An *n-quasitrace* on A is 1-quasitrace on A that extends to a 1-quasitrace on $A \otimes M_n$.

Proposition 1.4.4. *Any lower-semicontinuous 2-quasitrace on A extends to a lower-semicontinuous 1-quasitrace on $A\otimes\mathcal{K}$. Hence, it has been commonly used in the literature that $QT_2(A)$ is the set of all lower-semiconituous quasitraces on $A \otimes \mathcal{K}$, or equivalently, the set of all 2-quasitraces on A.*
(Note that it is also of common use to denote $QT(A)$ the set of all 1-quasitraces on A.)

Remark 1.4.5. Since we consider positive elements of $A\otimes\mathcal{K}$ in the context of Cu-semigroups, the interesting objects are the 2-quasitraces on A. By our discussion above we have: $T(A) \subseteq QT_2(A) \subseteq QT(A)$.

Definition 1.4.6. Let A be a C^*-algebra. A *functional* on $\mathrm{Cu}(A)$ is a generalized Cu-morphism $\alpha : \mathrm{Cu}(A) \longrightarrow \overline{\mathbb{R}}_+$. We denote by $F(\mathrm{Cu}(A))$ the set of all functionals on $\mathrm{Cu}(A)$.

Definition 1.4.7. A *cone* is an abelian monoid endowed with an $\mathbb{R}_+^*$-multiplication.

Theorem 1.4.8. *[30, Theorem 4.4]*
(i) $T(A)$, $QT_2(A)$ and $F(\mathrm{Cu}(A))$ are cones and moreover their scalar multiplication can be extended to $\overline{\mathbb{R}}_+$.
(ii) $QT_2(A) \simeq F(\mathrm{Cu}(A))$ as $\overline{\mathbb{R}}_+$-cones. Indeed for any $\tau \in QT_2(A)$, one can consider the following functional:

$$\begin{aligned} d_\tau : \mathrm{Cu}(A) &\longrightarrow \overline{\mathbb{R}}_+ \\ [a] &\longmapsto \sup_{n\in\mathbb{N}} \tau(a^{1/n}) \end{aligned}$$

and for any $\lambda \in F(\mathrm{Cu}(A))$, one can consider the following 1-quasitrace on $A \otimes \mathcal{K}$:

$$\begin{aligned} \tau_\lambda : (A \otimes \mathcal{K})_+ &\longrightarrow \overline{\mathbb{R}}_+ \\ a &\longmapsto \int_0^\infty \lambda([(a-t)_+])dt \end{aligned}$$

Finally the maps that send $\tau \longmapsto d_\tau$ and $\lambda \longmapsto \tau_\lambda$ are inverses of one another.

Theorem 1.4.9. *(Haagerup [39, Theorem 5.11] - see e.g. [30, Remark 4.3])*
Whenever A is exact, we have $T(A \otimes \mathcal{K}) = QT_2(A)$. Using Proposition 1.4.2, we deduce that whenever A is exact, $T(A) \simeq F(\mathrm{Cu}(A))$ as $\overline{R}_+$-cones.

Corollary 1.4.10. *Let us denote by C^*_{ex} the category of separable exact C^*-algebra and by* Cone *the category of whose objects are $\overline{R}_+$-cones and morphisms are monoid morphisms that preserve the $\overline{R}_+$-multiplication.*
Then T and $F \circ \mathrm{Cu}$ are well-defined continuous contravariant functors from C^*_{ex} to Cone*. Besides there exists a natural isomorphism between them. That is, $F \circ \mathrm{Cu} \simeq T$*

Proof. The continuity of the functors is proved in [30, Theorem 4.8]. Even though this natural isomorphism is a direct consequence of everything explained before, it is not explicitly stated in [30], so we will prove it here. Let $A \in C^*_{ex}$. We have seen that there exists $\eta_A : T(A) \simeq F(\mathrm{Cu}(A))$ that sends $\tau \longmapsto d_\tau$. Let $\phi : A \longrightarrow B$. By contravariance of the functors T and $F \circ \mathrm{Cu}$ we get the following maps:

$$\begin{aligned} T(\phi) : T(B) &\longrightarrow T(A) \\ \tau &\longmapsto \tau \circ \phi \end{aligned} \qquad\qquad \begin{aligned} F(\mathrm{Cu}(\phi) : F(\mathrm{Cu}(B)) &\longrightarrow F(\mathrm{Cu}(A)) \\ \lambda &\longmapsto \lambda \circ \mathrm{Cu}(\phi) \end{aligned}$$

And since $\tau \circ \phi(a^{1/n}) = \tau(\phi(a)^{1/n})$ for any $\tau \in T(B)$, any $a \in A_+$, we get that $d_{\tau\circ\phi} = d_\tau \circ \mathrm{Cu}(\phi)$. In other words, the following square is commutative:

$$\begin{array}{ccc} T(B) & \xrightarrow{\eta_B} & F(\mathrm{Cu}(B)) \\ {\scriptstyle T(\phi)}\downarrow & & \downarrow{\scriptstyle F\circ\mathrm{Cu}(\phi)} \\ T(A) & \xrightarrow[\eta_A]{} & F(\mathrm{Cu}(A)) \end{array}$$

which ends the proof. □

Definition 1.4.11. Let A be a C^*-algebra. Let us denote by $\mathrm{Ban}_{\mathbb{R}}$ the category whose objects are real Banach spaces, that is, a partially ordered Banach vector space over $\mathbb{R}$ and morphisms are real Banach space, that is, continuous $\mathbb{R}$-linear maps that respect the order.
Now let us consider $\mathrm{Aff}\, T(A) := \{\text{Continuous affine maps } f : T(A) \longrightarrow \mathbb{R}\}$. Then $\mathrm{Aff} : \mathrm{Cone} \longrightarrow \mathrm{Ban}_{\mathbb{R}}$ is a continuous contravariant functor. Hence $\mathrm{Aff}\, T : C^*_{ex} \longrightarrow \mathrm{Ban}_{\mathbb{R}}$ is a continuous covariant functor.

Chapter 2

The Cu_1 semigroup

In this chapter, we are going to define a new invariant for C^*-algebras. As stated before, the aim is to 'merge ' the Cuntz semigroup and the K_1-group. The first section is about defining the invariant, its categorical setting and describe its first properties. The second section and third section are focused on the continuity of our invariant. Finally, in the last section, we give an analogous notion to real rank zero in the categorical setting of the invariant.
To ease the notations, we will in this chapter use C^* to denote the category of separable C^*-algebras of stable rank one.

2.1 Definition of the invariant and its first properties

2.1.1. In this section, we will define our new invariant, the Cu_1-semigroup, and describe its first properties. Let us start with an important lemma and the definition of a preorder in order to get all the tools we need to do so.

Lemma 2.1.2. *Let A be a C^*-algebra with stable rank one and let a and b be contractions in A_+ such that $a \lesssim_{\mathrm{Cu}} b$. Let α and β be in A^{**} such that they both realize the Peligrad-Zsidó subequivalence of $p_a \lesssim_{PZ} p_b$ as in Definition 1.1.27. For any $u \in \mathcal{U}(\mathrm{her}\, a^\sim)$, we have*

$$[\theta^\sim_{ab,\alpha}(u)]_{\mathrm{K}_1(\mathrm{her}\, b^\sim)} = [\theta^\sim_{ab,\beta}(u)]_{\mathrm{K}_1(\mathrm{her}\, b^\sim)}$$

where $\theta^\sim_{ab,\alpha}$ (resp $\theta^\sim_{ab,\beta}$) is the unitized morphism of $\theta_{ab,\alpha}$ in Proposition 1.1.28.

Proof. In this proof, since a and b are fixed elements, we will only write θ_α for $\theta_{ab,\alpha}$ (resp θ_β for $\theta_{ab,\beta}$).

We consider the two injections given by α and β:

$$\begin{aligned} \theta_\alpha : \operatorname{her} a &\hookrightarrow \operatorname{her} b \\ d &\longmapsto \alpha^* d\alpha \end{aligned} \qquad \begin{aligned} \theta_\beta : \operatorname{her} a &\hookrightarrow \operatorname{her} b \\ d &\longmapsto \beta^* d\beta \end{aligned}$$

We define $x := a^{1/2}\alpha$ and $y := a^{1/2}\beta$. We first consider elements of aAa. Later, we will use the continuity of these maps and that every element of $\operatorname{her} a$ can be approximate by elements of aAa to obtain our result. We can rewrite θ_α and θ_β as follows:

$$\begin{aligned} \theta_\alpha : aAa &\hookrightarrow bAb \\ a\delta a &\longmapsto x^* a^{1/2}\delta a^{1/2} x \end{aligned} \qquad \begin{aligned} \theta_\beta : aAa &\hookrightarrow bAb \\ a\delta a &\longmapsto y^* a^{1/2}\delta a^{1/2} y \end{aligned}$$

Let u be a unitary element of $\operatorname{her} a^\sim$. There exists a pair (u_0, λ) with $u_0 \in \operatorname{her} a$ and $\lambda \in \mathbb{C}$ such that $u = u_0 + \lambda$.
Let $\epsilon > 0$. Now let δ_ϵ be in A such that $\|u_0 - a\delta_\epsilon a\| < \epsilon$.
Since $a = xx^* = yy^*$, by [19, Lemma 2] we know there exists a unitary element u_ϵ of $\operatorname{her} b^\sim$ such that $\|y - xu_\epsilon\| < \epsilon$ (equivalently $\|u_\epsilon^* x^* - y^*\| < \epsilon$).
Now, we compute:

$$\begin{aligned} \|u_\epsilon^*\theta_\alpha^\sim(a\delta_\epsilon a + \lambda)u_\epsilon - \theta_\beta^\sim(a\delta_\epsilon a + \lambda)\| &= \|u_\epsilon^* x^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon - y^* a^{1/2}\delta_\epsilon a^{1/2} y\| \\ &\le \|u_\epsilon^* x^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon - y^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon\| \\ &\quad + \|y^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon - y^* a^{1/2}\delta_\epsilon a^{1/2} y\| \\ &\le \|u_\epsilon^* x^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon - y^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon\| \\ &\quad + \|y^* a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon - y^* a^{1/2}\delta_\epsilon a^{1/2} y\| \\ &\le \|u_\epsilon^* x^* - y^*\|\, \|a^{1/2}\delta_\epsilon a^{1/2} x u_\epsilon\| + \|y - xu_\epsilon\|\, \|a^{1/2}\delta_\epsilon a^{1/2}\|\, \|y^*\|. \end{aligned}$$

Since $a^2 \le a$, by [29, Lemma A.1] we know there exists a sequence $(z_n)_n$ in A such that $a^{1/2} = \lim_{n\in\mathbb{N}} az_n$, where all z_n are contractions. Equivalently, $a^{1/2} = \lim_{n\in\mathbb{N}} z_n^* a$. So we can find an k_ϵ in $\mathbb{N}$ such that $\|a^{1/2}\delta_\epsilon a^{1/2}\| \le \|z_{k_\epsilon}^* a\delta_\epsilon a z_{k_\epsilon}\| + \epsilon \le \|a\delta_\epsilon a\| + \epsilon$.
Hence, we get that $\|a^{1/2}\delta_\epsilon a^{1/2}\| \le \|a\delta_\epsilon a\| + \epsilon$. Moreover, $\|u_0\| - |\lambda| \le \|u_0 + \lambda\| = 1$. So we obtain that $\|u_0\| \le 2$, which leads to $\|a\delta_\epsilon a\| \le 2 + 2\epsilon \le 3$.
Putting all together, we finally get that $\|u_\epsilon^*\theta_\alpha^\sim(a\delta_\epsilon a + \lambda)u_\epsilon - \theta_\beta^\sim(a\delta_\epsilon a + \lambda)\| \le 6\epsilon$, which proves us that

$$u_\epsilon^*\theta_\alpha^\sim(a\delta_\epsilon a + \lambda)u_\epsilon \sim_h \theta_\beta^\sim(a\delta_\epsilon a + \lambda).$$

Now combining $u \sim_h a\delta_\epsilon a + \lambda$ with the continuity of $\theta_\alpha^\sim$ and $\theta_\beta^\sim$, we conclude:

$$u_\epsilon^* \theta_\alpha^\sim(u) u_\epsilon \sim_h \theta_\beta^\sim(u)$$

and the result follows. □

Definition 2.1.3. (The $\lesssim_1$ binary relation)
Let A be a C^*-algebra with *stable rank one*. Let $a, b \in A_+$ and let u, v be unitary elements of $\operatorname{her} a^\sim$ and $\operatorname{her} b^\sim$ respectively. We write $(a, u) \lesssim_1 (b, v)$ if:

$$\begin{cases} a \lesssim_{\mathrm{Cu}} b \\ [\theta_{ab,\alpha}^\sim(u)] = [v] \text{ in } \mathrm{K}_1(\operatorname{her} b^\sim) \end{cases}$$

where $\theta_{ab,\alpha}$ is the injection given by a partial isometry α as constructed in Proposition 1.1.28.

Lemma 2.1.4. *The relation $\lesssim_1$ is reflexive and transitive.*

Proof. Reflexivity of $\lesssim_1$ follows from the fact that $\lesssim_{\mathrm{Cu}}$ is reflexive and that $id_{\operatorname{her} a} = \theta_{aa,p_a}$.
Now let a, b and c be in A_+ and let u_a, u_b and u_c be unitary elements of $\operatorname{her} a^\sim$, $\operatorname{her} b^\sim$ and $\operatorname{her} c^\sim$ respectively. Assume that $(a, u_a) \lesssim_1 (b, u_b)$ and $(b, u_b) \lesssim_1 (c, u_c)$. By hypothesis, we know that $a \lesssim_{\mathrm{Cu}} b$ and $b \lesssim_{\mathrm{Cu}} c$. Since A has stable rank one, there exist $x, y \in A$ such that $a = xx^*$, $b = yy^*$, $x^*x \in \operatorname{her} b$ and $y^*y \in \operatorname{her} c$.
Let us consider the polar decompositions of x and y. That is, $x = a^{1/2}\alpha, y = b^{1/2}\beta$, for some partial isometries α, β of A^{**}. Using Proposition 1.1.28, we get $p_a = \alpha\alpha^* \sim_{PZ} \alpha^*\alpha \leq p_b$ and also $p_b \sim_{PZ} \beta^*\beta \leq p_c$. We set $q_a := \alpha^*\alpha, q_b := \beta^*\beta$. One can check that $\gamma := \alpha\beta$ is a partial isometry of A^{**}. Furthermore, $p_a = \gamma\gamma^*$.
Let us write $z := a^{1/2}\gamma$. Observe that $zz^* = a$ and also $z = x\beta$. We hence compute that $z^*z = \beta^* x^* x\beta \in \operatorname{her} c$. We deduce that $zz^* = a$ and $z^*z \in \operatorname{her} c$.
By [6, Proposition 2.12] we may write $x := u^*(x^*x)^{1/3}$ for some element u of A. Since $(x^*x) \in A_{p_b}$ and $\beta^* A_{p_b} \subseteq A$, we deduce that $\beta^* x^*$ is in A, and hence $z \in A$.
Using Proposition 1.1.31, we obtain that $q_c := \gamma^*\gamma$ is the support projection of z^*z and is Peligrad-Zsidó equivalent to p_a. Finally, Lemma 2.1.2 tells us that $\theta_{ac,\gamma} := \theta_{bc,\beta} \circ \theta_{ab,\alpha}$ is one of the morphism described in Proposition 1.1.28, from which the transitivity of $\lesssim_1$ follows. □

Remark 2.1.5. Let A be a C^*-algebra with stable rank one and let $a \in A_+$. We have seen that for any unitary element u of $\operatorname{her} a^\sim$ and any partial isometry $\alpha \in A^{**}$ such that $p_a = \alpha\alpha^*$, the K_1-class $[\theta_{ab,\alpha}^\sim(u)]$ does not depend on the α chosen. In the sequel, whenever $a \lesssim_{\mathrm{Cu}} b$,

we will refer to the maps $\theta^{\sim}_{ab,\alpha}$ as *standard maps* and will rewrite them as θ_{ab}, for obvious notational purposes. In particular, whenever $a \leq b$ observe that the canonical inclusion map i is a standard map.
Also, notice that every standard morphism between a and b gives rise to the same group morphism at the K_1-level, that we will denote by χ_{ab}. That is, $\chi_{ab} := \mathrm{K}_1(\theta_{ab}) : \mathrm{K}_1(\mathrm{her}\, a) \longrightarrow \mathrm{K}_1(\mathrm{her}\, b)$.

2.1.6. (The Cu_1-semigroup)
Let A be a C^*-algebra with *stable rank one*. We consider a set consisting of pairs of positive elements and unitaries as follows:

$$H(A) := \{(a,u) : a \in (A \otimes \mathcal{K})_+, u \in \mathcal{U}(\mathrm{her}\, a^{\sim})\}$$

By antisymetrizing $\lesssim_1$, we define an equivalence relation on $H(A)$, that we write as $\sim_1$. Now define:

$$\mathrm{Cu}_1(A) := H(A)/{\sim_1}$$

The equivalence class of an element $(a,u) \in H(A)$ is denoted by $[(a,u)]$.
By the isomorphism $\psi : M_2(A \otimes \mathcal{K}) \simeq A \otimes \mathcal{K}$ (see Definition 1.3.2), given any two elements $(a,u),(b,v) \in H(A)$, we know that $a \oplus b := \psi(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})$ is a positive element of $A \otimes \mathcal{K}$. Besides, $(\begin{smallmatrix} \mathrm{her}\, a & 0 \\ 0 & \mathrm{her}\, b \end{smallmatrix}) \subseteq \mathrm{her}(a \oplus b)$ and hence $u \oplus v := (\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix})$ is a unitary element of $\mathrm{her}(a \oplus b)^{\sim}$.

Thus, for any two $[(a,u)],[(b,v)] \in \mathrm{Cu}_1(A)$, we say $[(a,u)] \leq [(b,v)]$ if, $(a,u) \lesssim_1 (b,v)$ and we set $[(a,u)] + [(b,v)] := [(a \oplus b),(u \oplus v)]$.

Proposition 2.1.7. *Let A be a C^*-algebra with stable rank one. Then* $(\mathrm{Cu}_1(A),+,\leq)$ *defined in Paragraph 2.1.6 is a partially ordered monoid, whose neutral element is* $[(0_A, 1_{\mathbb{C}})]$.

Proof. By construction $\leq$ is a well-defined partial order on $\mathrm{Cu}_1(A)$. Further, given a,b in $(A \otimes \mathcal{K})_+$, we know that $a \oplus b$ is a positive element of $A \otimes \mathcal{K}$ and $u \oplus v$ is a unitary element of $(\mathrm{her}(a \oplus b)^{\sim})$. We only have to check that addition defined in Paragraph 2.1.6 does not depend on the representative chosen:
Let $x,y \in \mathrm{Cu}_1(A)$. Let $[(a_1,u_1)],[(a_2,u_2)]$ be representatives of x and $[(b_1,v_1)],[(b_2,v_2)]$ be representatives of y. Since $[a_1] = [a_2]$ and $[b_1] = [b_2]$ in $\mathrm{Cu}(A)$, we get that $[a_1 \oplus b_1] = [a_2 \oplus b_2]$ in $\mathrm{Cu}(A)$. We also know that $\chi_{a_1a_2}([u_1]) = [u_2]$ in $\mathrm{K}_1(\mathrm{her}\, a_2)$ and $\chi_{b_1b_2}([v_1]) = [v_2]$ in $\mathrm{K}_1(\mathrm{her}\, b_2)$. Hence we get $\theta_{a_1a_2}(u_1) \oplus \theta_{b_1b_2}(v_1) \sim_h u_2 \oplus v_2$ in $\mathrm{her}(a_1 \oplus b_1)^{\sim}$.
A similar argument gives us $\theta_{a_2a_1}(u_2) \oplus \theta_{b_2b_1}(v_2) \sim_h u_1 \oplus v_1$ in $\mathrm{her}(a_2 \oplus b_2)^{\sim}$. We conclude that

$[(a_1 \oplus b_1, u_1 \oplus v_1)] = [(a_2 \oplus b_2, u_2 \oplus v_2)]$. That is, the addition is well-defined and obviously is commutative. Finally one can check that for any $[(a, u)] \in \mathrm{Cu}_1(A)$, $[(a, u)] + [(0_A, 1_{\mathbb{C}})] = [(a, u)]$ and that $+$ and $\leq$ are compatible. □

Remark 2.1.8. $(\mathrm{Cu}_1(A), +, \leq) \notin \mathrm{PoM}$ in general, since some elements might not be positive.

2.1.9. We now proceed to establish certain properties of the semigroup $\mathrm{Cu}_1(A)$. In particular, we consider a *compact-containment* relation as in the Cu-semigroup; see Definition 1.3.7. Note that as for Cu, this auxiliary relation is entirely determined by the order. Also, we show that $(\mathrm{Cu}_1(A), \leq)$ satisfies the Cuntz axioms mentioned in Definition 1.3.2.

Definition 2.1.10. Let $(S, \leq)$ be an ordered monoid. For any s, t in S, we say that s is *way-below* t and we write $s \ll t$ if, for any increasing sequence $(z_n)_{n\in\mathbb{N}}$ that has a supremum in S such that $\sup\limits_{n\in\mathbb{N}} z_n \geq t$, there exists k such that $z_k \geq s$. This is an auxiliary relation on S (in the sense of Definition 1.3.6 except that we do not require that $0 \prec s$ for all $s \in S$) called the *compact-containment relation*. In particular $s \ll t$ implies $s \leq t$.

Lemma 2.1.11. *Let A be a C^*-algebra with stable rank one and let $a \in A_+$. For any $n \in \mathbb{N}$ write $a_n := (a - 1/n)_+$. Then:*
(i) $([a_n])_n$ is a $\ll$-increasing sequence in $\mathrm{Cu}(A)$ whose supremum is $[a]$.
(ii)

$$\mathrm{AbGp} - \varinjlim(\mathrm{K}_1(\mathrm{her}\, a_n), \chi_{a_n a_m}) \simeq (\mathrm{K}_1(\mathrm{her}\, a), \chi_{a_n a}).$$

Proof. (i) It is well-known that $[(a - \epsilon)_+] \ll [a]$ for any $\epsilon > 0$; see e.g [71, Proposition 2.61]
(ii) Observe that $a_n \leq a_m$ in A for any $n \leq m$ and hence the standard morphisms $\theta_{a_n a_m}$ and $\theta_{a_n a}$ are in fact canonical injections $i_{nm} : \mathrm{her}\, a_n \subseteq \mathrm{her}\, a_m$ and $i_{n\infty} : \mathrm{her}\, a_n \subseteq \mathrm{her}\, a$ with $\mathrm{her}\, a = \overline{\bigcup\limits_{n\in\mathbb{N}} \mathrm{her}\, a_n}$. We deduce that $\varinjlim\limits_{n}(\mathrm{her}\, a_n, \theta_{a_n a_m}) \simeq (\mathrm{her}\, a, \theta_{a_n a})$. The result follows by functoriality of the functor K_1. □

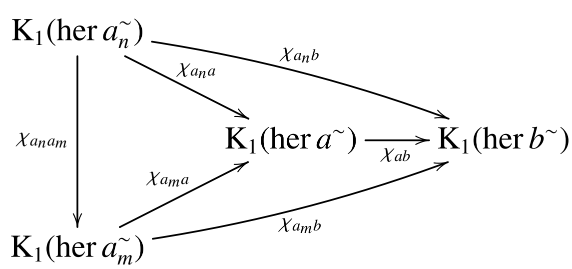
Proposition 2.1.12. *Let A be a C^*-algebra with stable rank one. Let $(a_n)_n$ be a sequence in A_+ such that $a_n \precsim_{\mathrm{Cu}} a_m$, for any $n \leq m$. Let $a \in A_+$ be any representative of $\sup\limits_n [a_n] \in \mathrm{Cu}(A)$ obtained from axiom (O1). Then for any unitary element $u \in \mathrm{her}\, a^\sim$, there exists a unitary element u_n in $\mathrm{her}\, a_n^\sim$ for some $n \in \mathbb{N}$ such that $[(a_n, u_n)] \leq [(a, u)]$ in $\mathrm{Cu}_1(A)$.*

Proof. For any $n \in \mathbb{N}$, consider $b_n := (a - 1/n)_+$. Using Lemma 2.1.11, we know that $[b_n] \ll [a]$ in $\mathrm{Cu}(A)$ for any $n \in \mathbb{N}$ and that $\mathrm{AbGp} - \varinjlim(\mathrm{K}_1(\mathrm{her}\, b_n), \chi_{b_n b_m}) \simeq (\mathrm{K}_1(\mathrm{her}\, a), \chi_{b_n a})$. Since we are in the category AbGp, the inductive limits are algebraic (see Paragraph 1.2.5).

Thus, for any $[u] \in \mathrm{K}_1(\mathrm{her}\, a)$, we can find $[u_n] \in K_1(\mathrm{her}\, b_n)$ such that $\chi_{b_n a}([u_n]) = [u]$. In fact, since A has stable rank one, then so does $\mathrm{her}\, b_n^\sim$. So by Proposition 1.1.17, we can find a unitary element u_n of $\mathrm{her}\, b_n^\sim$ whose K_1-class is $[u_n]$.
On the other hand, $([a_m])_m$ is an increasing sequence in $\mathrm{Cu}(A)$ whose supremum is $[a]$ and hence there exists $m \in \mathbb{N}$ such that $[b_n] \leq [a_m]$ in $\mathrm{Cu}(A)$. So we can consider the unitary element $\theta_{b_n a_m}(u_n)$ in $\mathrm{her}\, a_m^\sim$. By transitivity of $\precsim_1$, we obtain that $\chi_{a_m a}([\theta_{b_n a_m}(u_n)]) = \chi_{a_m a} \circ \chi_{b_n a_m}([u_n]) = \chi_{b_n a}([u_n]) = [u]$ and the result follows. □

Lemma 2.1.13. *Let A be a C^*-algebra with stable rank one. Then any increasing sequence in* $\mathrm{Cu}_1(A)$ *has a supremum.*

Proof. Let $([(a_n, u_n)])_{n\in\mathbb{N}}$ be an increasing sequence in $\mathrm{Cu}_1(A)$. Then $([a_n])_{n\in\mathbb{N}}$ is an increasing sequence in $\mathrm{Cu}(A)$. By (O1) in $\mathrm{Cu}(A)$, the sequence $([a_n])_{n\in\mathbb{N}}$ has a supremum $[a]$ in $\mathrm{Cu}(A)$. Now, let $n \leq m$. Since $[(a_n, u_n)] \leq [(a_m, u_m)]$, we get that $\chi_{a_n a_m}([u_n]) = [u_m]$. Besides, by transitivity of $\precsim_1$, we obtain that $\chi_{a_n a}([u_n]) = \chi_{a_m a}([u_m])$ in $\mathrm{K}_1(\mathrm{her}\, a)$. Write $[u] := \chi_{a_n a}([u_n])$. We deduce that $[(a, u)] \geq [(a_n, u_n)]$ in $\mathrm{Cu}_1(A)$, for any $n \in \mathbb{N}$.
Let us check that $[(a, u)]$ is in fact the supremum of the sequence. Let $[(b, v)] \in \mathrm{Cu}_1(A)$ such that $[(b, v)] \geq [(a_n, u_n)]$ for every $n \in \mathbb{N}$. Since $[a] = \sup\limits_{n\in\mathbb{N}}[a_n]$, we have $[b] \geq [a]$ in $\mathrm{Cu}(A)$. Using transitivity of $\precsim_1$, the following diagram is commutative:



Hence for every n and m in $\mathbb{N}$, we have $\chi_{a_n b}([u_n]) = \chi_{a_m b}([u_m]) = \chi_{ab}([u])$ in $\mathrm{K}_1(\mathrm{her}\, b)$. We deduce that $\chi_{ab}([u]) = [v]$ in $\mathrm{K}_1(\mathrm{her}\, b)$ and hence $[(a, u)] \leq [(b, v)]$. □

Proposition 2.1.14. *Let A be a C^*-algebra with stable rank one and let* $[(a, u)], [(b, v)] \in \mathrm{Cu}_1(A)$. *Then* $[(a, u)] \ll [(b, v)]$ *if and only if* $[a] \ll [b]$ *in* $\mathrm{Cu}(A)$ *and* $\chi_{ab}([u]) = [v]$ *in* $\mathrm{K}_1(\mathrm{her}\, b)$.

Proof. Suppose that $[(a, u)] \ll [(b, v)]$. A fortiori $[(a, u)] \leq [(b, v)]$, so $\chi_{ab}[u] = [v]$. Now let $([c_n])_n$ be an increasing sequence in $\mathrm{Cu}(A)$ whose supremum $[c]$ satisfies $[c] \geq [b]$. Write $w := \theta_{bc}(v)$ and consider $s := [(c, w)] \in \mathrm{Cu}_1(A)$. By Proposition 2.1.12, we know that there exists a unitary element w_n of her $c_n^{\sim}$ for some $n \in \mathbb{N}$ such that $\chi_{c_n c}([w_n]) = [w]$. Now define $s_k := [c_{n+k}, \theta_{c_n c_{n+k}}(w_n)]$, then $(s_k)_k$ is an increasing sequence in $\mathrm{Cu}_1(A)$. By the description of suprema obtained in Lemma 2.1.13, we know that $(s_k)_k$ admits s as a supremum. Plus, $s \geq [(b, v)]$ and since $[(a, u)] \ll [(b, v)]$, we deduce that there exists $k \in \mathbb{N}$ such that $[(a, u)] \leq s_k$ and hence that $[a] \leq [c_{n+k}]$. We conclude that $[a] \ll [b]$ in $\mathrm{Cu}(A)$.

Conversely, let $[(a, u)], [(b, v)] \in \mathrm{Cu}_1(A)$ such that $[a] \ll [b]$ in $\mathrm{Cu}(A)$ and $\chi_{ab}[u] = [v]$ in $\mathrm{K}_1(\mathrm{her}\, b)$. Now let $([(c_n, w_n)])_n$ be an increasing sequence in $\mathrm{Cu}_1(A)$ that has a supremum in $\mathrm{Cu}_1(A)$ that we write $[(c, w)]$. Also suppose that $[(b, v)] \leq [(c, w)]$. First, by transitivity of $\precsim_1$, observe that $\chi_{ac}([u]) = \chi_{bc} \circ \chi_{ab}([u]) = [w]$ in $\mathrm{K}_1(\mathrm{her}\, c)$.
Then, arguing as in the proof of [17, Lemma 4.3], since A has stable rank one, we can find a strictly decreasing sequence $(\epsilon_n)_n$ in $\mathbb{R}_+^*$ and unitary elements $(u_n)_n$ in $(A \otimes \mathcal{K})^{\sim}$ such that

$$\mathrm{her}(c_1 - \epsilon_1)_+ \subseteq u_1(\mathrm{her}(c_2 - \epsilon_2)_+)u_1^* \subseteq ... \subseteq u_n...u_1(\mathrm{her}(c_{n+1} - \epsilon_{n+1})_+)u_1^*...u_n^* \subseteq ...$$

and such that $\sup\limits_n [(c_n - \epsilon_n)_+] = [c]$ in $\mathrm{Cu}(A)$.
Hence, by Proposition 2.1.12 we can find a unitary element $\tilde{w_k}$ of $(\mathrm{her}(c_k - \epsilon_k)_+)^{\sim}$ such that $\chi_{(c_k - \epsilon_k)_+ c_k}[\tilde{w_k}] = [w_k]$ in $\mathrm{K}_1(\mathrm{her}\, c_k)$, for every $k \in \mathbb{N}$. Now, using the same argument as in the proof of Lemma 2.1.11, we observe that

$$\mathrm{AbGp} - \varinjlim(\mathrm{K}_1(\mathrm{her}(c_n - \epsilon_n)_+), \chi_{(c_n - \epsilon_n)_+ (c_m - \epsilon_m)_+}) \simeq (\mathrm{K}_1(\mathrm{her}\, c), \chi_{(c_n - \epsilon_n)_+ c}).$$

On the other hand, since $[a] \ll [b] \leq \sup\limits_n [(c_n - \epsilon_n)_+]$, there exists some $l \in \mathbb{N}$ large enough such that $[a] \leq [(c_l - \epsilon_l)_+]$ in $\mathrm{Cu}(A)$. Without loss of generality, we choose $l \geq k$ (we can always go to bigger indices if necessary).
Finally, using again transitivity of $\precsim_1$, we have that $\chi_{(c_l - \epsilon_l)_+ c}([\tilde{w}_l]) = \chi_{c_l c} \circ \chi_{(c_l - \epsilon_l)_+ c_l}([\tilde{w}_l]) = [w] = \chi_{ac}([u]) = \chi_{(c_l - \epsilon_l)_+ c} \circ \chi_{a (c_l - \epsilon_l)_+}([u])$ in $\mathrm{K}_1(\mathrm{her}\, c)$. Since we are in the category AbGp, the inductive limits are algebraic (see Paragraph 1.2.5), and thus there exists some $l' \geq l$ such that $\chi_{(c_l - \epsilon_l)_+ (c_{l'} - \epsilon_{l'})_+}([\tilde{w}_l]) = \chi_{(c_l - \epsilon_l)_+ (c_{l'} - \epsilon_{l'})_+} \circ \chi_{(a c_l - \epsilon_l)_+}([u])$. Composing with $\chi_{(c_{l'} - \epsilon_{l'})_+ c_{l'}}$ on both sides, we finally obtain that $[w_{l'}] = \chi_{a c_{l'}}[u]$ and hence $[(a, u)] \leq [(c_{l'}, w_{l'})]$, which ends the proof. □

Corollary 2.1.15. *Let A be a C^*-algebra with stable rank one and let $[(a, u)] \in \mathrm{Cu}_1(A)$. Then $[(a, u)]$ is compact if and only if $[a]$ is compact in* $\mathrm{Cu}(A)$.

Theorem 2.1.16. *Let A be a C^*-algebra with stable rank one. Then $(\mathrm{Cu}_1(A), \leq)$ satisfies axioms (O1), (O2), (O3), and (O4).*

Proof. (O1) is done in Lemma 2.1.13.

(O2): Let $s := [(a,u)] \in \mathrm{Cu}_1(A)$. We want to write s as the supremum of a $\ll$-increasing sequence in $\mathrm{Cu}_1(A)$. By (O2), we can find a $\ll$-increasing sequence $([a_n])_n$ in $\mathrm{Cu}(A)$ such that $\sup\limits_n [a_n] = [a]$. Write a_n any representative of $[a_n]$ in $(A \otimes \mathcal{K})_+$. Using Proposition 2.1.12, we know that we can find a unitary element u_n of her $a_n^\sim$ for some $n \in \mathbb{N}$ such that $[(a_n, u_n)] \leq [(a,u)]$. Now we consider $s_k := [(a_{n+k}, \theta_{a_n a_{n+k}}(u_n))]$, for any $k \in \mathbb{N}$. Then, by Proposition 2.1.14 we deduce that $(s_k)_k$ is a $\ll$-increasing sequence in $\mathrm{Cu}_1(A)$. By the description of suprema obtained in Lemma 2.1.13, $\sup\limits_k s_k = s$.

(O3): Let $[(a_1,u_1)] \ll [(b_1,v_1)]$ and $[(a_2,u_2)] \ll [(b_2,v_2)]$. We already know that $[(a_1,u_1)] + [(a_2,u_2)] \leq [(b_1,v_1)] + [(b_2,v_2)]$ and that $[a_1] + [a_2] \ll [b_1] + [b_2]$ in $\mathrm{Cu}(A)$. The conclusion follows from Proposition 2.1.14.

(O4): Let $([(a_n,u_n)])_{n\in\mathbb{N}}$ and $([(b_n,v_n)])_{n\in\mathbb{N}}$ be two increasing sequences in $\mathrm{Cu}_1(A)$. Let $[(a,u)] := \sup\limits_{n\in\mathbb{N}} [(a_n,u_n)]$ and $[(b,v)] := \sup\limits_{n\in\mathbb{N}} [(b_n,v_n)]$. Now we define $([(c_n,w_n)])_{n\in\mathbb{N}} := ([(a_n,u_n)])_{n\in\mathbb{N}} + ([(b_n,v_n)])_{n\in\mathbb{N}}$. Since $[c_n] = [a_n] + [b_n]$ in $\mathrm{Cu}(A)$ and $\mathrm{Cu}(A)$ satisfies (O4), we have $\sup\limits_{n\in\mathbb{N}} [c_n] = [a \oplus b]$. Also, we know that $\chi_{a_n a}([u_n]) = [u]$ and $\chi_{b_n b}([v_n]) = [v]$, and hence we obtain $\chi_{c_n c}(u_n \oplus v_n) = u \oplus v$. We conclude that sup and + are compatible in $\mathrm{Cu}_1(A)$, using Lemma 2.1.13. □

2.1.17. We have proved that $\mathrm{Cu}_1(A)$ is a semigroup that satisfies the Cuntz axioms. The aim now is to define a functor Cu_1 from the category C^* to a suitable category of semigroups as was done for the Cu-semigroup; see [4, Chapter 3]. The category Cu would then be a natural candidate for the codomain category, however the underlying ordered monoid in $\mathrm{Cu}_1(A)$ is usually not positively ordered. Thus we have to find a suitable category and study its relation with the category Cu.

Definition 2.1.18. Let $\mathrm{PoM}^\sim$ be the category of ordered monoids, -not necessarily positively ordered-, whose morphisms are ordered monoid morphism. Now we define the category $\mathrm{Cu}^\sim$ as follows:

$\mathrm{Ob}(\mathrm{Cu}^\sim)$: Ordered monoids satisfying the Cuntz axioms and such that $0 \ll 0$.
$\mathrm{Morph}(\mathrm{Cu}^\sim)$: $\mathrm{PoM}^\sim$-morphisms that respect suprema of increasing sequences and the compact-containment relation.

2.1.19. We will see in another section of the chapter that the category $\mathrm{Cu}^\sim$ defined is indeed a

suitable one to define a continuous functor $\mathrm{Cu}_1 : C^* \longrightarrow \mathrm{Cu}^\sim$, where we recall that C^* denotes the full subcategory consisting of separable C^*-algebras with stable rank one. To do so, we use an analogous process as done in [4, Chapter 2 - Chapter 3] for the Cu-semigroup, and thus we are going to first consider a pre-completed version of Cu_1, that we will denote by W_1, to then extend the result to Cu_1 using Category Theory techniques.
Before going into details, let us end this section with a categorical link between $\mathrm{PoM}, \mathrm{Cu}$ and their respective analogous versions $\mathrm{PoM}^\sim, \mathrm{Cu}^\sim$.

Definition 2.1.20. Let $M \in \mathrm{PoM}^\sim$ and let $S \in \mathrm{Cu}^\sim$. We define their *positive cones*, that we write M_+ and S_+ respectively, as the subset of positive elements. Observe that $M_+ \in \mathrm{PoM}$ and $S_+ \in \mathrm{Cu}$.

Lemma 2.1.21. *Let C be either* PoM *or* Cu. *Then C is a coreflective subcategory of $C^\sim$. A fortiori it is a full subcategory. Also, the following functor:*

$$\begin{aligned} \nu_+ : C^\sim &\longrightarrow C \\ S &\longrightarrow S_+ \\ f &\longrightarrow f_+ \end{aligned}$$

is a coreflector. (See Paragraph 1.2.4.)

Proof. Since $C^\sim$-morphisms respect $\leq$, we deduce that ν_+ is a well-defined functor. Moreover, one can check that $Hom_{C^\sim}(i(S), T) \simeq Hom_C(S, \nu_+(T))$ for any $S \in C$ and $T \in C^\sim$. We get that the inclusion functor $i : C \hookrightarrow C^\sim$ is left adjoint to ν_+, which implies that C is a full (obviously faithful) coreflective subcategory of $C^\sim$. □

2.2 A pre-completed version of Cu$_1$: W$_1$

2.2.1. In this section, we define a precompleted version of our invariant, that we write W_1, in a slightly more general categorical context. For that matter, we will introduce categories analogous to PreW and W introduced in [4, Chapter 2] that we will call $\mathrm{PreW}^\sim$ and $\mathrm{W}^\sim$. Finally, even if all the following is adapted to our setting, we will refer each time the reader to its analogous version in [4].

Definition 2.2.2. Let $S \in \mathrm{PoM}^\sim$ An *auxiliary relation* on, S is a binary relation $\prec$ such that:
(i) For any $s, t \in S$ such that $s \prec t$, then $s \leq t$.
(ii) For any $s, t, s', t' \in S$ such that $s \leq s' \prec t' \leq t$, then $s \prec t$.

Observe that $\prec$ is transitive. Also the compact-containment relation in Cu_1 is an auxiliary relation, with the additional feature that it is entirely determined by the order.

2.2.3. (cf [4, §2.1.1])
Let $S \in \mathrm{PoM}^{\sim}$ and consider an auxiliary relation $\prec$ on S. For any $s \in S$ we denote $s_{\prec} := \{s' \in S \mid s' \prec s\}$. Let us introduce the W-axioms:
(W1): For any $s \in S$, there exists a $\prec$-increasing sequence $(s_k)_k$ in $s_{\prec}$ such that for any $s' \in s_{\prec}$, there exists some k such that $s' \prec s_k$.
(W2): For any $s \in S$, we have $s = \sup s_{\prec}$.
(W3): Addition and $\prec$ are compatible.
(W4): For any $s, t, x \in S$ such that $x \prec s + t$, we can find $s', t' \in S$ such that $s' \prec s, t' \prec t$ and $x \prec s' + t'$.

Definition 2.2.4. (cf [4, Definition 2.1.2])
A $\mathrm{PreW}^{\sim}$-*semigroup* is a pair $(S, \prec)$, where $S \in \mathrm{PoM}^{\sim}$ and $\prec$ is an auxiliary relation on S such that $(S, \prec)$ satisfies axioms (W1)-(W3)-(W4) and such that $0 \prec 0$.
If moreover $(S, \prec)$ satisfies (W2), we say it is a $\mathrm{W}^{\sim}$-*semigroup*. Whenever the context is clear, we omit the reference to $\prec$ and simply write $S \in \mathrm{PreW}^{\sim}$.

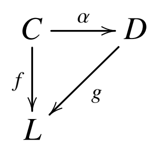
A $\mathrm{W}^{\sim}$-*morphism* between any two $S, T \in \mathrm{PreW}^{\sim}$ is a $\mathrm{PoM}^{\sim}$-morphism $g : S \longrightarrow T$ that respects the auxiliary relation and satisfies the following $\mathrm{W}^{\sim}$-*continuity axiom*:
(M) For any $s \in S$ and $t \in T$ such that $t \prec g(s)$, there exists $s' \in s_{\prec}$ such that $t \leq g(s')$.

Finally, we define the categories $\mathrm{PreW}^{\sim}$ and $\mathrm{W}^{\sim}$ whose objects are respectively $\mathrm{PreW}^{\sim}$-semigroups and $\mathrm{W}^{\sim}$-semigroups and whose morphisms are $\mathrm{W}^{\sim}$-morphisms. Observe that $\mathrm{W}^{\sim}$ is a full subcategory of $\mathrm{PreW}^{\sim}$.

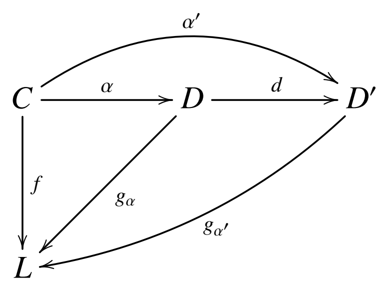
2.2.5 (Completion of $\mathrm{W}^{\sim}$)**.** The next step is to show that $\mathrm{W}^{\sim}$ is in fact a reflective subcategory of $\mathrm{PreW}^{\sim}$. That is, the inclusion functor $\mathrm{W}^{\sim} \hookrightarrow \mathrm{PreW}^{\sim}$ has a left-adjoint (see Paragraph 1.2.4). We will use a categorical tool described in [45]: *the completion property*.

Definition 2.2.6. Let $\mathcal{C}$ be category and $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. Let $C \in \mathcal{C}$. A $\mathcal{D}$-*completion of* C is a couple (D, α), where $D \in \mathcal{D}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, such that the following holds:
For any $L \in \mathcal{D}$, $f \in \mathrm{Hom}_{\mathcal{C}}(C, L)$, there exists a unique $g \in \mathrm{Hom}_{\mathcal{D}}(D, L)$ such that the following

diagram is commutative:



Proposition 2.2.7. *[45, Definition 2.1] A $\mathcal{D}$ − completion (D, α) of $C \in \mathcal{C}$ is unique (up to isomorphism). That is, if (D', α') is another $\mathcal{D}$ − completion of C then there exists a unique $\mathcal{D}$-isomorphism $d : D \longrightarrow D'$ such that the following diagram is commutative:*



2.2.8. The definition below is equivalent to the one in Paragraph 1.2.4.

Definition 2.2.9. (Reflective subcategory - Completion property) [45, Definition 2.1]
Let $\mathcal{C}$ be a category. Then $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$ if and only if for any $C \in \mathcal{C}$, there exists a $\mathcal{D}$ − *completion* of C.

Proposition 2.2.10. *(cf [4, Proposition 2.1.6])* $\mathrm{W}^\sim$ *is a (full) reflective subcategory of* $\mathrm{PreW}^\sim$.

Proof. Let $(S, \prec)$ be a $\mathrm{PreW}^\sim$ semigroup. We will construct a $\mathrm{W}^\sim$-completion of S. Let s, t be in S. We recall that we denote $s_\prec := \{s' \in S \mid s' \prec s\}$. Consider the binary relation on S given by $s \preceq t$ if $s_\prec \subseteq t_\prec$. By antisymmetrizing it, we obtain an equivalence relation $\sim$ on S. That is, $s \sim t$ if, $s \preceq t$ and $t \preceq s$. Now, we consider $\mu^\sim(S) := S/\sim$. We define a partial order on $\mu(S)$ given by $[s] \leq [t]$ if, $s \preceq t$ and we define addition on equivalence classes induced by the addition on S. We finally get that $(\mu^\sim(S), +, \leq) \in \mathrm{PoM}^\sim$. For $s, t \in S$, we define $[s] \prec [t]$ in $\mu^\sim(S)$ if there exists $t' \in t_\prec$ such that $[s] \leq [t']$. Observe that $[0] \prec [0]$.
Arguing as in [4, Proposition 2.1.6], we obtain $\mu^\sim(S) \in \mathrm{W}^\sim$. Let $\eta_S : S \longrightarrow \mu^\sim(S)$ be the $\mathrm{W}^\sim$-morphism that assigns to each element in S its equivalence class. Then $(\mu^\sim(S), \eta_S)$ is a $\mathrm{W}^\sim$-completion of S. We conclude, using Definition 2.2.9, that $\mu^\sim$ is a reflector from $\mathrm{PreW}^\sim$ to $\mathrm{W}^\sim$. □

2.2.11. The next step is to build direct limits in $\mathrm{PreW}^\sim$ using the algebraic limit in $\mathrm{PoM}^\sim$ which is the same as for PoM (see Paragraph 1.2.5). This will allow us to compute inductive limits in $\mathrm{W}^\sim$ through the reflector $\mu^\sim$ obtained in the proof of Proposition 2.2.10.

Definition 2.2.12. Let $(S_i, \varphi_{ij})_{i\in I}$ be an inductive system in $\mathrm{PreW}^{\sim}$. We define $S := \mathrm{PoM}^{\sim} - \varinjlim(S_i, \varphi_{ij})_{i\in I}$ constructed as in Paragraph 1.2.5. We define an auxiliary relation $\prec$ on S as follows: Let $s, t \in S$. We say $s \prec t$ if, $\varphi_{ik}(s_i) \prec \varphi_{jk}(t_j)$, where $s_i \in S_i$, $t_j \in S_j$ are representatives of s, t respectively and $k \geq i, j$.

Proposition 2.2.13. *(i) The category* $\mathrm{PreW}^{\sim}$ *has finite sums.*
(ii) The category $\mathrm{PreW}^{\sim}$ *has inductive limits. More precisely, let* $(S_i, \varphi_{ij})_{i\in I}$ *be an inductive system in* $\mathrm{PreW}^{\sim}$ *and let* $S := \mathrm{PoM}^{\sim} - \varinjlim(S_i, \varphi_{ij})$*. Then* $(S, \prec) \simeq \mathrm{PreW}^{\sim} - \varinjlim(S_i, \varphi_{ij})$*, where* $\prec$ *is constructed as in Definition 2.2.12.*

Proof. (i) One can define component-wise addition, order and auxiliary relation.
(ii) The proof is virtually the same as in [4, Theorem 2.1.8] and we obtain that $\prec$ is an auxiliary relation on $(S, \prec)$ that satisfies $(W1), (W3)$ and $(W4)$. Finally, since $0_{S_i} \prec 0_{S_i}$ for any i, we trivially have $0_S \prec 0_S$. □

Corollary 2.2.14. $\mathrm{W}^{\sim}$ *has inductive limits. Moreover:*

$$\mathrm{W}^{\sim} - \varinjlim(S_i, \varphi_{ij}) = \mu^{\sim}(\mathrm{PreW}^{\sim} - \varinjlim(S_i, \varphi_{ij}))$$

2.2.15. (Local C^*-algebras)
Now that we have a well-defined categorical setup, we will define a functor from C^*_{loc} to $\mathrm{W}^{\sim}$, termed W_1, and show that it is continuous. First let us recall some definitions and properties about C^*_{loc}. We refer the reader to [4, §2.2] for more details.

A *local* C^**-algebra* A is an upward-directed union of C^*-algebras. That is, there exists a family of complete *-invariant subalgebras $\{A_i\}_i$ such that for any i, j, there exists $k \geq i, j$ such that $A_i \cup A_j \subseteq A_k$ and $A = \bigcup_i A_i$.
If A is a local C^*, then so is $M_k(A)$ for any $k \in \mathbb{N}$. In fact, as done for C^*-algebra, $M_k(A)$ sits as upper-left corner inside $M_{k'}(A)$ for any $k' \geq k$ and we can picture any $M_k(A)$ as a corner of $M_\infty(A) := \bigcup_k M_k(A)$, which is again a local C^*-algebra.
Observe that the completion of a local C^*-algebra A, that we write $\overline{A}$, is a C^*-algebra. In particular, we have $\overline{M_k(A)} \simeq M_k(\overline{A})$ for any $k \in \mathbb{N}$ and $\overline{M_\infty(A)} \simeq \overline{A} \otimes \mathcal{K}$. Further A is closed under functional calculus.
Moreover, for any local C^*-algebra $A := \bigcup_i A_i$, if each A_i has stable rank one, then by [60, Theorem 5.1], we get that $\overline{A}$ has stable rank one. We may abuse the language and say that A has stable rank one.

We now consider C^*_{loc}, the category whose objects are separable local C^*-algebras that have stable rank one and morphisms are *-homomorphisms. Obviously, C^* is a full subcategory of C^*_{loc}. In fact, C^* is a reflective subcategory of C^*_{loc} with the following reflector:

$$\begin{aligned} \gamma : C^*_{loc} &\longrightarrow C^* \\ A &\longmapsto \overline{A} \\ \phi &\longmapsto \overline{\phi} \end{aligned}$$

where $\overline{\phi}$ is the (unique) extension of ϕ over $\overline{A}$.

Finally, let $(A_i, \varphi_{ij})_{i\in I}$ be an inductive system in C^*_{loc}. As in [4, §2.2.8], we consider the algebraic inductive limit $A_{alg} := \bigsqcup_{i\in I} A_i/\sim$ (see Paragraph 1.2.5) with the following pre-norm: $\|x\| := \inf_j\{\|\varphi_{ij}(x)\|\}$), for $x \in A_i$. We now define:

$$C^*_{loc} - \varinjlim(A_i, \varphi_{ij}) := (A_{alg}/N, \|\,\|)$$

where $N := \{a \in A_{alg}, \|a\| = 0\}$.

Besides, φ_{ij} induces a *-homomorphism that we also write $\varphi_{ij} : M_\infty(A_i) \longrightarrow M_\infty(A_j)$ and we have $C^*_{loc} - \varinjlim(M_\infty(A_i), \varphi_{\tilde{ij}}) \simeq M_\infty(C^*_{loc} - \varinjlim(A_i, \varphi_{ij}))$. See [4, §2.2.8].

Remark 2.2.16. A notion of *Bass stable rank* was first introduced for unital rings, (see [7, Definition 4.0], [60, Proposition 2.2]) and by [40, Theorem], one could conjecture that a C^*_{loc}-algebra A has Bass stable rank one if and only if its completion $\overline{A}$ has stable rank one.

2.2.17. (The precompleted Cuntz semigroup $\mathrm{W}(A)$)

We briefly recall the definition of the precompleted Cuntz semigroup of a C^*-algebra and we refer the reader to [4, §2.2] for details and proofs. In fact, we give an equivalent definition that can be found in [4, Remark 3.2.4]; see also [4, Lemma 3.2.7].

Let $A \in C^*_{loc}$. We define $\mathrm{W}(A) := \{[a] \in \mathrm{Cu}(\overline{A}) \mid a \in M_\infty(A)_+\}$. Obviously, $(\mathrm{W}(A), +, \leq) \in$ PoM as a submonoid of $\mathrm{Cu}(\overline{A})$. Now we equip $\mathrm{W}(A)$ with the following auxiliary relation. Given $[a], [b] \in \mathrm{W}(A)$, we write $[a] \prec [b]$ if:

$$a \lesssim_{\mathrm{Cu}} (b-\epsilon)_+ \text{ in } M_\infty(A)_+ \text{ for some } \epsilon > 0.$$

We have that $(\mathrm{W}(A), \prec) \in \mathrm{W}^\sim$. (See [4, Proposition 2.2.5].)

Lemma 2.2.18. *Let $A \in C^*_{loc}$ and let $B := \overline{A}$ be its completion in C^*. Then, for any $a \in A_+$ we have $\overline{aAa} = \overline{aBa}$.*

Proof. The direct inclusion is trivial. Now let $x \in \overline{aBa}$. Then there exists a sequence $(b_k)_k$ in B such that $x = \lim_k ab_k a$. Furthermore, for any $k \in \mathbb{N}$, there exists a sequence $(a_{k,i})_i$ in A such that $b_k = \lim_i a_{k,i}$. We deduce that $x = \lim_k a(\lim_i a_{k,i})a = \lim_k \lim_i (aa_{k,i}a)$. Thus $x \in \overline{aAa}$. □

Definition 2.2.19. Let $A \in C^*_{loc}$ and let $B := \overline{A}$ be its completion in C^*. For $a \in A_+$, we define the *hereditary subalgebra generated by* a as $\mathrm{her}\, a := \overline{aBa}$.

2.2.20. We have now all the tools to define a precompleted version of Cu_1 that we will denote by $\mathrm{W}_1(A)$, as a submonoid of $\mathrm{Cu}_1(\overline{A})$.

Definition 2.2.21. Let $A \in C^*_{loc}$. We define $\mathrm{W}_1(A) := \{[(a,u)] \in \mathrm{Cu}_1(\overline{A}) \mid a \in M_\infty(A)_+\}$. Obviously, $(\mathrm{W}_1(A), +, \leq) \in \mathrm{PoM}^\sim$ as a submonoid of $\mathrm{Cu}_1(\overline{A})$. Now we equip $\mathrm{W}_1(A)$ with the following binary relation. Let $[(a,u)], [(b,v)] \in \mathrm{W}_1(A)$, we say $[(a,u)] \prec [(b,v)]$ if:

$$\begin{cases} a \lesssim_{\mathrm{Cu}} (b-\epsilon)_+ \text{ in } M_\infty(A)_+ \text{ for some } \epsilon > 0. \\ [\theta_{ab}(u)] = [v] \text{ in } \mathrm{K}_1(\mathrm{her}\, b^\sim). \end{cases}$$

Remark 2.2.22. Let $A \in C^*$ and let $[(a,u)], [(b,v)] \in \mathrm{Cu}_1(A)$. Then $[(a,u)] \prec [(b,v)]$ if and only if $[a] \prec [b]$ in $\mathrm{W}(A)$ and $\chi_{ab}([u]) = [v]$ in $\mathrm{K}_1(\mathrm{her}\, b)$.

Lemma 2.2.23. *Let $A \in C^*_{loc}$ and let $a \in A_+$. For any $n \in \mathbb{N}$ write $a_n := (a - 1/n)_+$. Then:*
(i) $([a_n])_n$ is a $\prec$-increasing sequence in $\mathrm{W}(A)$. Besides, $([a_n])_n$ has a supremum in $\mathrm{W}(A)$ and $\sup_n [a_n] = [a]$.
(ii)

$$\mathrm{AbGp} - \varinjlim(\mathrm{K}_1(\mathrm{her}\, a_n), \chi_{a_n a_m}) \simeq (\mathrm{K}_1(\mathrm{her}\, a), \chi_{a_n a}).$$

Proof. Combine the fact that $\overline{A}$ has stable rank one, with Definition 2.2.19 and the result follows from Lemma 2.1.11. □

Corollary 2.2.24. *Let $A \in C^*_{loc}$. Let $(a_n)_n$ be a sequence in A_+ such that $a_n \lesssim_{\mathrm{Cu}} a_m$ for any $n \leq m$. Also we suppose that $[a_n]_n$ has a supremum in $W(A)$ that we write $[a]$. Let $a \in A_+$ be any representative of* $\sup_n [a_n] \in \mathrm{W}(A)$. *Then for any unitary element $u \in \mathrm{her}\, a^\sim$, there exists a unitary element u_n in $\mathrm{her}\, a_n^\sim$ for some $n \in \mathbb{N}$ such that $[(a_n, u_n)] \leq [(a,u)]$ in $\mathrm{W}_1(A)$.*

Proof. Again, combine the fact that $\overline{A}$ has stable rank one, with Definition 2.2.19 and the result follows from Proposition 2.1.12. □

Proposition 2.2.25. *[4, Proposition 2.2.5]*
Let $A \in C^*_{loc}$. The relation defined in Definition 2.2.21 is an auxiliary relation and $(W_1(A), \prec)$ satisfies axioms (W1),(W2),(W3) and (W4). That is, $(W_1(A), \prec) \in \mathrm{W}^\sim$. Again, we may omit the reference to $\prec$ and simply write $\mathrm{W}_1(A) \in \mathrm{W}^\sim$.

Proof. Using Remark 2.2.22, it follows that $\prec$ is an auxiliary relation on $W_1(A)$. Namely, if $[(a,u)] \leq [(b,v)] \prec [(c,w)] \leq [(d,z)]$, then we have $\chi_{ad}([u]) = [z]$ and $a \lesssim_{\mathrm{Cu}} (d-\epsilon)_+$ for some $\epsilon > 0$ since $b \lesssim_{\mathrm{Cu}} (c-\delta)_+$ for some $\delta > 0$. Thus, $[(a,u)] \prec [(d,z)]$.
Now, given $[(a,u)] \in \mathrm{W}_1(A)$, use Lemma 2.2.23 to construct a sequence in $\mathrm{W}_1(A)$ where $[((a-1/n)_+, u_n)] \prec [(a,u)]$ and in such a way that $[(a,u)] = \sup\limits_n [((a-1/n)_+, u_n)]$; see Lemma 2.1.13. Thus (W2) holds in $\mathrm{W}_1(A)$.
If $[(b,v)] \prec [(a,u)]$, then, by Proposition 2.1.14, we have $[(b,v)] \ll [(a,u)]$ in $\mathrm{Cu}_1(A)$ and thus $[(b,v)] \leq [((a-1/n)+, u_n)]$ for some $n \in \mathbb{N}$. Hence (W2) holds. To check (W3) and (W4) is routine. □

Proposition 2.2.26. *Let $\varphi : A \longrightarrow B$ be a *-homomorphism between $A, B \in C^*_{loc}$. Observe that φ naturally extends to a *-homomorphism $\varphi : M_\infty(A) \longrightarrow M_\infty(B)$ (we use the same notation). We write $\overline{\varphi} := \gamma(\varphi)$ and $\overline{\varphi}^\sim$ the unitized morphism between $\overline{M_\infty(A)}^\sim \longrightarrow \overline{M_\infty(B)}^\sim$. Then the map:*

$$\begin{aligned} \mathrm{W}_1(\varphi) : \mathrm{W}_1(A) &\longrightarrow \mathrm{W}_1(B) \\ [(a,u)] &\longmapsto [(\varphi(a), \overline{\varphi}^\sim(u))] \end{aligned}$$

is a $\mathrm{W}^\sim$-morphism.

Proof. Let $a \in M_\infty(A)_+$. Using Paragraph 1.1.4, the restriction $\overline{\varphi}_{|\,\mathrm{her}\, a} : \mathrm{her}\, a \longrightarrow \mathrm{her}\, \varphi(a)$ of φ gives us the following commutative square:

$$\begin{array}{ccc} \mathrm{her}\, a & \xrightarrow{\ \overline{\varphi}\ } & \mathrm{her}(\varphi(a)) \\ \downarrow & & \downarrow \\ \mathrm{her}\, a^\sim & \xrightarrow[\ \overline{\varphi}^\sim\ ]{} & (\mathrm{her}\, \varphi(a))^\sim \end{array}$$

Hence, $\overline{\varphi}^\sim(u)$ is a unitary element of $(\mathrm{her}\, \varphi(a))^\sim$ and we deduce that $[(\varphi(a), \overline{\varphi}^\sim(u))] \in \mathrm{W}_1(B)$. Let us check it does not depend on the representative (a,u) chosen. Let $[(a,u)], [(b,v)] \in \mathrm{W}_1(A)$ such that $[(a,u)] \leq [(b,v)]$. Then we get $a \lesssim_{\mathrm{Cu}} b$ in $M_\infty(A)$. Since φ is a *-homomorphism, we deduce that $\varphi(a) \lesssim_{\mathrm{Cu}} \varphi(b)$ in $M_\infty(B)$. Further, using [57, Theorem 26.55], one can show that for any partial isometry α of $(A \otimes \mathcal{K})^{**}$ that realizes one of our standard morphisms $\theta_{ab,\alpha}$ (see Remark 2.1.5) between her a and her b, then $\overline{\varphi}^{**}(\alpha)$ is a partial isometry of $(B\otimes\mathcal{K})^{**}$ that realizes one of our standard morphisms $\theta_{\varphi(a)\varphi(b),\overline{\varphi}^{**}(\alpha)}$ between her $\varphi(a)$ and $\varphi(b)$, where $\overline{\varphi}^{**}:(A\otimes\mathcal{K})^{**}\longrightarrow(B\otimes\mathcal{K})^{**}$ is a *-homomorphism induced by $\overline{\varphi}$. It follows that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{her}\, a^{\sim} & \xrightarrow{\theta_{ab,\alpha}} & \mathrm{her}\, b^{\sim} \\ {\scriptstyle \overline{\varphi}^{\sim}}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}^{\sim}} \\ (\mathrm{her}\,\varphi(a))^{\sim} & \xrightarrow[\theta_{\varphi(a)\varphi(b),\overline{\varphi}^{**}(\alpha)}]{} & (\mathrm{her}\,\varphi(b))^{\sim} \end{array}$$

from which we deduce that $\theta_{\varphi(a)\varphi(b)}(\overline{\varphi}^{\sim}(u))\sim\overline{\varphi}^{\sim}(v)$ and thus $[(\varphi(a),\overline{\varphi}^{\sim}(u))]\leq[(\varphi(b),\overline{\varphi}^{\sim}(v))]$. So $\mathrm{W}_1(\varphi)$ is indeed well-defined, respects $\leq$ and it is easy to check that $\mathrm{W}_1(\varphi)$ also respects addition. We conclude that $\mathrm{W}_1(\varphi)$ is a $\mathrm{PoM}^{\sim}$-morphism.

Since φ is a *-homomorphism, we have that $\varphi((a-\epsilon)_+)=(\varphi(a)-\epsilon)_+$ for any $a\in\mathrm{M}_\infty(A)_+$ and any $\epsilon>0$.
Now, let $[(a,u)],[(b,v)]\in\mathrm{W}_1(A)$ such that $[(a,u)]\prec[(b,v)]$. That is, there exists $\epsilon>0$ such that $[a]\leq[(b-\epsilon)_+]$ in $\mathrm{W}(A)$ and $\chi_{ab}([u])=[v]$ in $\mathrm{K}_1(\mathrm{her}\, b)$. Then using the above and continuity of φ, we obtain that $[\varphi(a)]\leq[(\varphi(b)-\epsilon 1_{B^{\sim}})_+]$ in $\mathrm{W}(B)$ and $\chi_{\varphi(a)\varphi(b)}([\overline{\varphi}^{\sim}(u)])=[\overline{\varphi}^{\sim}(v)]$ in $\mathrm{K}_1(\mathrm{her}\,\varphi(b))$. We conclude that $\mathrm{W}_1(\varphi)$ also respects $\prec$.

Further, we have to check that $\mathrm{W}_1(\varphi)$ satisfies the $\mathrm{W}^{\sim}$-continuity axiom (see Definition 2.2.4). Let us write $f:=\mathrm{W}_1(\varphi)$. Let $x:=[(a,u)]\in\mathrm{W}_1(A)$ and $y:=[(b,v)]\in\mathrm{W}_1(B)$ such that $y\prec f(x)$. We have to find $x'\in\mathrm{W}_1(A)$ such that $x'\prec x$ and $y\leq f(x')$.
We know that there exists $k>0$ such that $[b]\leq[(\varphi(a)-1/k)_+]$ in $\mathrm{W}(B)$ and $\chi_{b\varphi(a)}([v])=[\overline{\varphi}^{\sim}(u)]$ in $\mathrm{K}_1(\mathrm{her}\,\varphi(a))$. On the other hand, observe that $([(a-1/n)_+])_n$ is an increasing sequence in $\mathrm{W}(A)$ that has admits $[a]$ as supremum in $\mathrm{W}(A)$. Thus, by Corollary 2.2.24, we can find a unitary element $u_n\in\mathrm{her}((a-1/n)_+)^{\sim}$ for some $n\in\mathbb{N}$, such that $[((a-1/n)_+,u_n)]\leq[(a,u)]$ in $\mathrm{W}_1(A)$. In particular, $[((\varphi(a)-1/m)_+,\overline{\varphi}^{\sim}(\theta_{(a-1/n)_+(a-1/m)_+}(u_n)))]\leq[(\varphi(a),\overline{\varphi}^{\sim}(u))]$ in $\mathrm{W}_1(B)$ for any $m\geq n$.
Now choose $m>k,n$, we get the following:

$$\begin{cases} [b]\leq[(\varphi(a)-1/k)_+]\leq[(\varphi(a)-1/m)_+] \text{ in } \mathrm{W}(B). \\ [\theta_{b\varphi(a)}(v)]=[\theta_{(\varphi(a)-1/n)_+\varphi(a)}(\overline{\varphi}^{\sim}(u_n))] \text{ in } \mathrm{K}_1(\mathrm{her}\,\varphi(a)). \end{cases}$$

By transitivity of $\lesssim_1$, we obtain:

$$[\theta_{(\varphi(a)-1/m)_+\varphi(a)} \circ \theta_{b(\varphi(a)-1/m)_+}(v)] = [\theta_{(\varphi(a)-1/m)_+\varphi(a)} \circ \theta_{(\varphi(a)-1/n)_+(\varphi(a)-1/m)_+}(\overline{\varphi}^{\sim}(u_n))] \text{ in } \mathrm{K}_1(\mathrm{her}\,\varphi(a)).$$

Finally, by Lemma 2.2.23, we conclude that there exists $l \geq m$ such that:

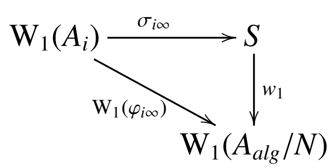
$$\begin{cases} [b] \leq [(\varphi(a) - 1/l)_+] \text{ in } \mathrm{W}(B). \\ [\theta_{b(\varphi(a)-1/l)_+}(v)] = [\theta_{(\varphi(a)-1/n)_+(\varphi(a)-1/l)_+}(\overline{\varphi}^{\sim}(u_n))] \text{ in } \mathrm{K}_1(\mathrm{her}(\varphi(a) - 1/l)_+). \end{cases}$$

Write $x' := [((a - 1/l)_+, \theta_{(a-1/n)_+(a-1/l)_+}(u_n))]$. Then we already know that $x' \prec x$ in $\mathrm{W}_1(A)$ and the above exactly states that $y \leq f(x')$ in $\mathrm{W}_1(B)$. □

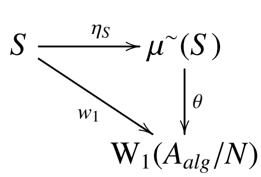
Corollary 2.2.27. *The assignment* $A \longmapsto \mathrm{W}_1(A)$ *from* C^*_{loc} *to* $\mathrm{W}^{\sim}$ *is a functor.*

Theorem 2.2.28. *The functor* $\mathrm{W}_1 : C^*_{loc} \longrightarrow \mathrm{W}^{\sim}$ *is continuous.*

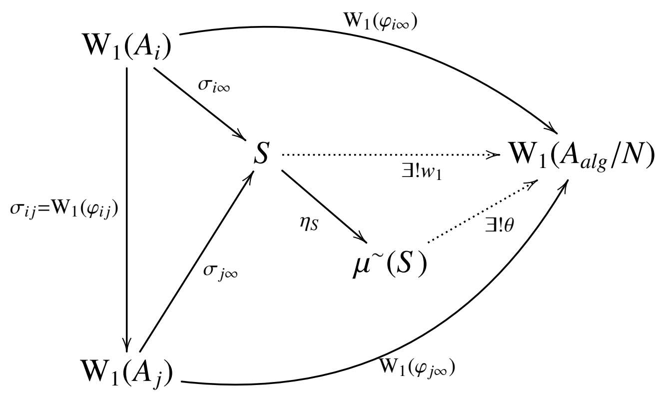
Proof. This proof is an adapted version of [4, Theorem 2.2.9].
Let $(A_i, \varphi_{ij})_{i\in I}$ be an inductive system in C^*_{loc} and let $(A_{alg}/N, \varphi_{i\infty})$ be its inductive limit. Without loss of generality, we can suppose that each $A_i \simeq M_\infty(A_i)$; see Paragraph 2.2.15. Thus, we may suppose that each element of $\mathrm{W}(A_i)$ is realized by a positive element of A_i.
Let $\sigma_{ij} := \mathrm{W}_1(\varphi_{ij})$ and consider the inductive system $(\mathrm{W}_1(A_i), \psi_{ij})_{i\in I}$ in $\mathrm{PreW}^{\sim}$. We denote by $(S, \sigma_{i\infty})$ its inductive limit in $\mathrm{PreW}^{\sim}$. Observe that $(\mathrm{W}_1(A_{alg}/N), \mathrm{W}_1(\varphi_{i\infty}))$ is a cocone for the inductive system; see Paragraph 1.2.3. Hence from universal properties, we deduce that there exists a unique $w_1 : S \longrightarrow \mathrm{W}_1(A_{alg}/N)$ such that for all $i \in I$, the following diagram commutes:



Moreover we know from Proposition 2.2.10 that $(\mu^{\sim}(S), \eta_S)$ is the inductive limit in $\mathrm{W}^{\sim}$ of the inductive system above, where $\eta_S : S \longrightarrow \mu^{\sim}(S)$. Hence, there exists a unique $\theta : \mu^{\sim}(S) \longrightarrow \mathrm{W}_1(A_{alg}/N)$ such that the following diagram commutes:



Let us sum up the context with the following commutative diagram:



To complete the proof, let us show that θ is a $\mathrm{W}^{\sim}$-isomorphism.

Surjectivity: Let $[(a,u)] \in \mathrm{W}_1(A_{alg}/N)$. Since $a \in A_{alg}/N$, we know that there exists $a_k \in (A_k)_+$ such that $\varphi_{k\infty}(a_k) = a$. Also, u is a unitary element of her $a^{\sim} = \overline{\varphi_{k\infty}(a_k)(A_{alg}/N)\varphi_{k\infty}(a_k)}^{\sim}$. Now, observe that $C^* - \varinjlim_{j>k}(\mathrm{her}\,\varphi_{kj}(a_k), \varphi_{jl}) \simeq (\mathrm{her}\,a, \varphi_{j\infty})$. Hence for any $\epsilon > 0$, there exists $j \geq k$ and a unitary element u_j of her $\varphi_{kj}(a_k)^{\sim}$ such that $\|u - \overline{\varphi}^{\sim}_{j\infty}(u_j)\| < \epsilon$. In particular, for $\epsilon < 2$, we obtain a unitary element u_j of her $\varphi_{kj}(a_k)^{\sim}$ such that $[u] = [\overline{\varphi}^{\sim}_{j\infty}(u_j)]$ in $\mathrm{K}_1(\mathrm{her}\,a)$.
We compute that $\mathrm{W}_1(\varphi_{j\infty})([(\varphi_{kj}(a_k), u_j)]) = [(\varphi_{k\infty}(a_k), \overline{\varphi_{j\infty}}^{\sim}(u_j))] = [(a,u)]$. Thus, by the commutativity of the diagram above we obtain $w_1 \circ \sigma_{j\infty}([(\varphi_{kj}(a_k), u_j)]) = \mathrm{W}_1(\varphi_{j\infty})([(\varphi_{kj}(a_k), u_j)]) = [(a,u)]$ as desired. We conclude that w_1 is surjective and hence that θ is surjective.

Injectivity: Let us show that for any $s, t \in S$ such that $w_1(s) \leq w_1(t)$ then $s \leq t$. In fact, it is sufficient to prove that for any $s, t \in S$ such that $w_1(s) \leq w_1(t)$ then $s^{\prec} \subseteq t^{\prec}$. Indeed this would imply that $\eta_S(s) \leq \eta_S(t)$, and since $\mathrm{im}(\eta_S) = \mu^{\sim}(S)$, we would conclude that θ is order-embedding.
Let $s, t \in S$ such that $w_1(s) \leq w_1(t)$ and let $s' \prec s$. Since the inductive limit is algebraic (see Proposition 2.2.13), there exists $s_k, s'_k, t_k \in W_1(A_k)$ such that $\sigma_{k\infty}(s'_k) = s', \sigma_{k\infty}(s_k) = s$ and $\sigma_{k\infty}(t_k) = t$ and such that $s'_k \prec s_k$ in $W_1(A_k)$.
Now pick $a', a, b \in (A_k)_+$ and unitary elements u', u, v in the respective hereditary subalgebras such that $s'_k = [(a', u')], s_k = [(a,u)]$ and $t_k = [(b,v)]$. We already know that $[a'] \prec [a]$ in $\mathrm{W}(A_k)$ and that $[\theta_{a'a}(u')] = [u]$ in $\mathrm{K}_1(\mathrm{her}\,a^{\sim})$. On the other hand, since $w_1(s) \leq w_1(t)$, by the

commutativity of the diagram, we deduce that:

$$\begin{cases} [\varphi_{k\infty}(a')] \prec [\varphi_{k\infty}(a)] \leq [\varphi_{k\infty}(b)] \text{ in } \mathrm{W}(A_{alg}/N). \\ [\theta_{\varphi_{k\infty}(a')\varphi_{k\infty}(b)}(\overline{\varphi_{k\infty}}^{\sim}(u'))] = [\theta_{\varphi_{k\infty}(a')\varphi_{k\infty}(b)}(\overline{\varphi_{k\infty}}^{\sim}(u))] = [\overline{\varphi_{k\infty}}^{\sim}(v)] \text{ in } \mathrm{K}_1(\mathrm{her}\,\varphi_{k\infty}(b)). \end{cases}$$

By the proof [4, Theorem 2.2.9], we deduce that there exists some $j \geq k$ and some $\delta > 0$ such that:

$$[\varphi_{kj}(a')] \leq [(\varphi_{kj}(b) - \delta)_+] \text{ in } \mathrm{W}(A_j).$$

Finally, by Lemma 2.2.23, we conclude that there exists $l \geq k, j$ such that:

$$\begin{cases} [\varphi_{kl}(a')] \leq [(\varphi_{kl}(b) - \delta)_+] \text{ in } \mathrm{W}(A_l). \\ [\theta_{\varphi_{kl}(a')\varphi_{kl}(b)}(\overline{\varphi_{kl}}^{\sim}(u'))] = [\overline{\varphi_{kl}}^{\sim}(v)] \text{ in } \mathrm{K}_1(\mathrm{her}\,\varphi_{kl}(b)). \end{cases}$$

We conclude that $\sigma_{kl}(s'_k) \prec \sigma_{kl}(t_k)$, which ends the proof. □

2.3 The functor Cu_1

2.3.1. Now that we have shown the continuity of the functor $\mathrm{W}_1 : C^*_{loc} \longrightarrow \mathrm{W}^{\sim}$, we will show that $\mathrm{Cu}^{\sim}$ is a full reflexive subcategory of $\mathrm{W}^{\sim}$. We will then get inductive limits in $\mathrm{Cu}^{\sim}$. Further, we define a functor $\mathrm{Cu}_1 : C^* \longrightarrow \mathrm{Cu}^{\sim}$, and prove that it is continuous with respect to inductive limits. We will adapt arguments from [4, Chapter 3].

Lemma 2.3.2. *(cf [4, Lemma 3.1.4])*
Let $f : M \longrightarrow N$ be a $\mathrm{PoM}^{\sim}$ − morphism between two $\mathrm{Cu}^{\sim}$-semigroups. Then the following are equivalent:
(i) f preserves suprema of increasing sequences.
(ii) f satisfies the $\mathrm{W}^{\sim}$-continuity axiom.
(iii) For any $a \in S$, we have $f(a) = \sup f(a^{\ll})$.

Proof. The proof is virtually the same as in [4, Lemma 3.1.4] as it does not use the fact that the underlying monoids are positively ordered. □

Corollary 2.3.3. $\mathrm{Cu}^{\sim}$ *is a full subcategory of* $\mathrm{W}^{\sim}$.

Proposition 2.3.4. *(cf [4, Proposition 3.1.6])*
Let $(S, \prec)$ be a $\mathrm{PreW}^{\sim}$-semigroup. Then there exists a $\mathrm{Cu}^{\sim}$-semigroup $\gamma^{\sim}(S)$ together with a $\mathrm{W}^{\sim}$-morphism $\alpha_S : S \longrightarrow \gamma^{\sim}(S)$ satisfying the following conditions:

(i) The morphism α_S is an 'auxiliary-embedding' in the sense that $s' \prec s$ whenever $\alpha(s') \ll \alpha(s)$.
(ii) The morphism α_S has a 'dense image' in the sense that for any two $t', t \in \gamma(S)$ such that $t' \ll t$ there exists $s \in S$ such that $t' \leq \alpha(s) \leq t$.

Proof. The proof is virtually the same as in [4, Proposition 3.1.6] as it does not use the fact that the underlying monoids are positively ordered. We still give a sketch of the proof for the sake of completeness.
Let $(S, \prec)$ be a $\mathrm{PreW}^{\sim}$-semigroup. Consider $\overline{S} := \{\prec\text{-increasing sequences in } S\}$. We denote such sequences by $\overline{a} := (a_n)_{n\in\mathbb{N}}$. Now for any two $\prec$-increasing sequences $\overline{a}, \overline{b}$ of $\overline{S}$, we write that $\overline{a} \subset \overline{b}$ if for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $a_k \prec b_n$. Also, we naturally define $\overline{a} + \overline{b} := (a_n + b_n)_n$ in $\overline{S}$.
We antisymmetrize the relation $\subset$ to get an equivalence relation on $\overline{S}$ and we define $\gamma^{\sim}(S) := \bar{S}/\sim$. We get an induced addition on $\gamma^{\sim}(S)$. Besides, the relation $\subset$ on $\overline{S}$ induces a partial order on $\gamma^{\sim}(S)$. Finally the sequence $(0, 0, ...)$ is an element of $\overline{S}$ since $0 \prec 0$. Thus the set $(\gamma^{\sim}(S), \leq)$ belongs to $\mathrm{PoM}^{\sim}$, whose neutral element is $[\overline{0}]$.
Further, we refer the reader to the proof of [4, Proposition 3.1.6] to obtain that $(\gamma^{\sim}(S), \leq)$ satisfies axioms (O1), (O2), (O3), (O4).
Finally, let us define $\alpha : S \longrightarrow \gamma^{\sim}(S)$ as follows: For any $a \in S$, fix $\overline{a} := (a_n)_{n\in\mathbb{N}}$ one of the sequences that satisfies axiom (W1) and set $\alpha_S(a) := [\overline{a}]$. One can check that $\alpha_S(a)$ does not depend on the $(a_n)_{n\in\mathbb{N}}$ chosen and it is straightforward to check that α_S is an 'auxiliary-embedding' with 'dense image'. □

Theorem 2.3.5. *(cf [4, Theorem 3.1.8])*
Let S be a $\mathrm{PreW}^{\sim}$-semigroup and T a $\mathrm{Cu}^{\sim}$-semigroup. Let $\eta : S \longrightarrow T$ be a $\mathrm{W}^{\sim}$-morphism. Then the following are equivalent:
(i) η is an auxiliary-embedding and has a dense image as in Proposition 2.3.4.
(ii) (T, η) is a $\mathrm{Cu}^{\sim}$-completion of S in the sense of Definition 2.2.9.

Proof. The proof is virtually the same as in [4, Theorem 3.1.8] as it does not use the fact that the underlying monoids are positively ordered. □

Corollary 2.3.6. $\mathrm{Cu}^{\sim}$ *is a (full) reflective subcategory of* $\mathrm{PreW}^{\sim}$. *In particular,* $\mathrm{Cu}^{\sim}$ *has inductive limits.*

Proof. Combine Theorem 2.3.5 with Proposition 2.3.4 and use the completion property; see Definition 2.2.9. □

2.3.7. We have everything we need to properly define the functor Cu_1 and prove that it is continuous. We recall that we denote by C^* the full subcategory of separable C^*-algebras with stable rank one. Also, we recall that for $A \in C^*$, we have already defined $(\mathrm{Cu}_1(A), \leq)$ and proved that it is a $\mathrm{Cu}^\sim$-semigroup; see Theorem 2.1.16.
Finally, observe that for any $A \in C^*$, the compact-containment relation on $\mathrm{Cu}_1(A)$ and the auxiliary relation on $\mathrm{W}_1(A \otimes \mathcal{K})$ agree; see [4, Remark 3.2.4]. Thus, we have that $\mathrm{Cu}_1(A) = \mathrm{W}_1(A \otimes \mathcal{K})$ as $\mathrm{Cu}^\sim$-semigroups.

Proposition 2.3.8. *Let $\varphi : A \longrightarrow B$ a *-homomorphism between $A, B \in C^*$. Observe that φ naturally extends to a *-homomorphism that we also write $\varphi : A \otimes \mathcal{K} \longrightarrow B \otimes \mathcal{K}$. We denote by $\varphi^\sim$ the unitized morphism between $A \otimes \mathcal{K}^\sim \longrightarrow B \otimes \mathcal{K}^\sim$. Then:*

$$\begin{aligned} \mathrm{Cu}_1(\varphi) : \mathrm{Cu}_1(A) &\longrightarrow \mathrm{Cu}_1(B) \\ [(a,u)] &\longmapsto [(\varphi(a), \varphi^\sim(u))] \end{aligned}$$

is a $\mathrm{Cu}^\sim$*-morphism.*

Proof. Using the identification of Paragraph 2.3.7 combined with Proposition 2.2.26, we get the result. □

Theorem 2.3.9. *There exists a natural isomorphism $\gamma^\sim \circ \mathrm{W}_1 \simeq \mathrm{Cu}_1 \circ \gamma$, where γ is the reflector from C^*_{loc} to C^*, see Paragraph 2.2.15. In particular, for any $A \in C^*$, there is a (natural)* $\mathrm{Cu}^\sim$*-isomorphism between* $\mathrm{Cu}_1(A) \simeq \gamma^\sim(\mathrm{W}_1(A))$.

Proof. The aim of the proof is to show that $(\mathrm{Cu}_1(\gamma(A), W_1(i))$ is a $\mathrm{Cu}^\sim$-completion of $W_1(A)$ for any $A \in C^*_{loc}$, where $W_1(i)$ is built as follows:
Let $A \in C^*_{loc}$, write $B := M_\infty(A) \in C^*_{loc}$. Consider the canonical inclusion $i : B \hookrightarrow \overline{B} \simeq \overline{A} \otimes \mathcal{K}$. Then i induces a $\mathrm{W}^\sim$-morphism $W_1(i) : \mathrm{W}_1(B) \longrightarrow \mathrm{W}_1(\overline{B})$. On the other hand, we know that $\mathrm{W}_1(B) = \mathrm{W}_1(A)$ and that $\mathrm{W}_1(\overline{B}) \simeq \mathrm{Cu}_1(\overline{A})$ (see Paragraph 2.3.7). Thus, we obtain a $\mathrm{W}^\sim$-morphism $W(i) : \mathrm{W}_1(A) \longrightarrow \mathrm{Cu}_1(\overline{A})$ (we use the same notation). By Theorem 2.3.5, we only have to check that $W_1(i)$ is an auxiliary-embedding and that it has a dense image as defined in Proposition 2.3.4.
Let $s, s' \in \mathrm{W}_1(A)$ such that $\mathrm{W}_1(i)(s') \ll \mathrm{W}_1(i)(s')$. By Paragraph 2.3.7, we deduce that $\mathrm{W}_1(i)(s') \prec \mathrm{W}_1(i)(s')$. Also, observe that $\mathrm{W}_1(i)$ is in fact an order embedding (even more, it is the canonical injection). Thus, we conclude that $s \prec s'$ and hence $\mathrm{W}_1(i)$ is an 'auxiliary-embedding'.
Let $t, t' \in \mathrm{Cu}_1(\gamma(A))$ such that $t' \ll t$. Now pick $a, a' \in (\gamma(A) \otimes \mathcal{K})_+$ and unitary elements

u, u' in the respective hereditary subalgebras of a, a', such that $t := [(a, u)]$ and $t' := [(a', u')]$. Then, we know that $[a'] \ll [a]$ in $\mathrm{Cu}(\overline{A})$ and that $\chi_{a'a}([u']) = [u]$. Using [4, Lemma 3.2.7], there exists $b \in M_\infty(A)_+$ such that $[a'] \leq [b] \leq [a]$ in $\mathrm{Cu}(\overline{A})$. Now consider $s := [(b, \theta_{a'b}(u))] \in \mathrm{W}_1(A)$, we get that $t' \leq \mathrm{W}_1(i)(s) \leq t$ in $\mathrm{Cu}_1(\overline{A})$. It follows that $\mathrm{W}_1(i)$ has a 'dense image' and hence that $(\mathrm{W}_1(i), \mathrm{Cu}_1(\gamma(A)))$ is a $\mathrm{Cu}^\sim$-completion of $\mathrm{W}_1(A)$. □

Corollary 2.3.10. *The functor* $\mathrm{Cu}_1 : C^* \longrightarrow \mathrm{Cu}^\sim$ *is continuous. More precisely, given an inductive system* $(A_i, \phi_{ij})_{i\in I}$ *in* C^**, then:*

$$\mathrm{Cu}^\sim - \varinjlim(\mathrm{Cu}_1(A_i), \mathrm{Cu}_1(\phi_{ij})) \simeq \mathrm{Cu}_1(C^* - \varinjlim((A_i, \phi_{ij}))) \simeq \gamma^\sim(\mathrm{W}^\sim - \varinjlim(\mathrm{W}_1(A_i), \mathrm{W}_1(\phi_{ij}))).$$

2.4 Algebraic Cu˜-semigroups and PoM˜-completion

2.4.1. In this last section, we will briefly introduce algebraic $\mathrm{Cu}^\sim$-semigroups in order to link the notion of real rank zero for a C^*-algebra A, that ensures a lot of projections, with the notion of 'density' of compact elements in $\mathrm{Cu}_1(A)$. In fact, as compact elements of $\mathrm{Cu}_1(A)$ are entirely determined by to the ones of its positive cone $\mathrm{Cu}(A)$ (see Corollary 2.1.15), all results from $\mathrm{Cu}(A)$ will apply here. These can be found in [4, §5.5].

2.4.2. We have already defined a functor $\nu_+ : \mathrm{Cu}^\sim \longrightarrow \mathrm{Cu}$ (see Lemma 2.1.21) that associates to every $\mathrm{Cu}^\sim$-semigroup S its positive cone S_+. Let us do something similar to recover the compact elements of a $\mathrm{Cu}^\sim$-semigroup.

Lemma 2.4.3. *(i) Let* $S \in \mathrm{Cu}^\sim$*. We denote by* $S_c := \{s \in S \mid s \ll s\}$*. Observe that* $S_c \in \mathrm{PoM}^\sim$*.*
(ii) Let $f : S \longrightarrow T$ *be a* $\mathrm{Cu}^\sim$*-morphism between* $S, T \in \mathrm{Cu}^\sim$*. Observe that* $f(S_c) \subset T_c$*. Thus, we define a* $\mathrm{PoM}^\sim$*-morphism* $f_c := f_{|S_c} : S_c \longrightarrow T_c$*.*
Finally we define the following functor:

$$\begin{aligned} \nu_c : \mathrm{Cu}^\sim &\longrightarrow \mathrm{PoM}^\sim \\ S &\longmapsto S_c \\ f &\longmapsto f_c \end{aligned}$$

Proof. Since any $\mathrm{Cu}^\sim$-morphism sends compact elements to compact elements, and since any $\mathrm{Cu}^\sim$-morphism is in particular a $\mathrm{PoM}^\sim$-morphism, the result follows. □

Proposition 2.4.4. *Let* $M \in \mathrm{PoM}^\sim$*. Then* $(M, \leq) \in \mathrm{W}^\sim$*. We denote* $\mathrm{Cu}^\sim(M) := \gamma^\sim(M, \leq)$ *the* $\mathrm{Cu}^\sim$*-completion of* $(M, \leq)$ *(see Proposition 2.3.4). Any* $\mathrm{PoM}^\sim$*-morphism* $f : M \longrightarrow N$

between $M, N \in \mathrm{PoM}^\sim$ *induces a* Cu$^\sim$*-morphism* $\gamma^\sim(f) : \gamma^\sim(M) \longrightarrow \gamma^\sim(N)$*. Thus we obtain a functor:*

$$\begin{aligned} \mathrm{Cu}^\sim : \mathrm{PoM}^\sim &\longrightarrow \mathrm{Cu}^\sim \\ M &\longmapsto \mathrm{Cu}^\sim(M) \\ f &\longmapsto \gamma^\sim(f) \end{aligned}$$

Proof. Observe that in the case where the auxiliary relation is the same as the order, the completion process corresponds to 'adding' suprema of $\leq$-increasing sequences. Further, the induced morphism of f naturally sends suprema of $\leq$-increasing sequences of $\mathrm{Cu}^\sim(M)$ to the ones in $\mathrm{Cu}^\sim(N)$. See [4, §5.5.3] □

Definition 2.4.5. Let $S \in \mathrm{Cu}^\sim$. We say that S is an *algebraic* Cu$^\sim$*-semigroup* if every element in S is the supremum of an increasing sequence of compact elements, that is, an increasing sequence in S_c. We denote by $\mathrm{Cu}^\sim_{alg}$ the full subcategory of Cu$^\sim$ consisting of algebraic Cu$^\sim$-semigroups (see [4, §5.5]).

Proposition 2.4.6. *(cf [4, Proposition 5.5.4])*
(i) For any algebraic Cu$^\sim$*-semigroup* S*, we have* $\mathrm{Cu}^\sim(S_c) \simeq S$.
(ii) $\mathrm{Cu}^\sim(M)$ *is an algebraic* Cu$^\sim$*-semigroup for any* $M \in \mathrm{PoM}^\sim$.

Lemma 2.4.7. *[21, Corollary 5] Whenever A has real rank zero,* Cu(A) *is an algebraic* Cu*-semigroup. If moreover A has stable rank one, then the converse is true.*

Corollary 2.4.8. *Let* $A \in C^*$*. Then A has real rank zero if and only if* $\mathrm{Cu}_1(A) \in \mathrm{Cu}^\sim_{alg}$.

Proof. Using the characterization of compacts elements of $\mathrm{Cu}_1(A)$ by compact elements of $\mathrm{Cu}(A)$ as in Corollary 2.1.15, we get that $\mathrm{Cu}(A)$ is algebraic if and only if $\mathrm{Cu}_1(A)$ is algebraic. □

Remark 2.4.9. We end this section by observing that ν_+ and ν_c satisfy the following: $\nu_+ \circ \nu_c \simeq \nu_c \circ \nu_+$. Hence, we sometimes consider $\nu_{+,c} : \mathrm{Cu}^\sim \longrightarrow \mathrm{PoM}$ as the composition of ν_+ and ν_c. Naturally, for any $S \in \mathrm{Cu}^\sim$, we denote by $S_{+,c} := \nu_{+,c}(S)$ the positively ordered monoid of positive compact elements of S.

Chapter 3

The structure of the Cu_1-semigroup

The aim of this chapter is to study the ideal structure of $\mathrm{Cu}^\sim$-semigroups. The first section gives a new picture of the structure of the Cu_1-semigroup using the complete lattice of ideals of the C^*-algebra. In the second section, we define abstractly ideals in the category $\mathrm{Cu}^\sim$ and describe their properties. The last section is focused on the notion of quotient-ideals and short exact sequences in the category $\mathrm{Cu}^\sim$, to then apply all of the above to the Cu_1-semigroup.
As before, we shall assume that A is a separable C^*-algebra with has stable rank one. We recall that in order to ease the notations, we use C^* to denote the category of separable C^*-algebras of stable rank one.

3.1 Structure of the Cu_1-semigroup

3.1.1. The aim of this section is to use the ideal structure of the algebra. Notice that, unless specified, we only deal with closed two-sided ideals. We will show that the ideals of A can be useful to understand in a different way the Cu_1-semigroup of a C^*-algebra and morphisms between Cu_1-semigroups of C^*-algebras.

3.1.2. Let $A \in C^*$. Let a be a positive element in A. We will explicitly recall that her a and I_a are stably isomorphic and use those isomorphisms to rewrite $\mathrm{Cu}_1(A)$ in terms of ideals of A. First, we recall some results that can be found in [11], [13].

Theorem 3.1.3. *[11, Theorem 2.8] Let $A \in C^*$ and let B be a full hereditary subalgebra of A.*

Then there exists a partial isometry v *in* $\mathcal{M}(C \otimes \mathcal{K})$*, where* $C := \begin{pmatrix} B & \overline{B.A} \\ \overline{A.B} & A \end{pmatrix}$*, such that*

$$\begin{aligned} B \otimes \mathcal{K} &\simeq A \otimes \mathcal{K} \\ d &\longmapsto v^* d v \end{aligned}$$

Corollary 3.1.4. *Let* $A \in C^*$ *and let* B *be a full hereditary subalgebra of* A*. Then the canonical inclusion* $i : B \hookrightarrow A$ *induces a* **-isomorphism* $i \otimes 1_{M(\mathcal{K})} : B \otimes \mathcal{K} \simeq A \otimes \mathcal{K}$*.*

Proof. We have $(i \otimes 1_{M(\mathcal{K})})(B \otimes \mathcal{K}) \simeq (B \otimes \mathcal{K})$ which, by Theorem 3.1.3, is isomorphic to $A \otimes \mathcal{K}$. More generally, one can check that an injective *-homomorphism between isomorphic C^*-algebras is in fact an isomorphism. □

3.1.5. Let $A \in C^*$ and let $a \in (A \otimes \mathcal{K})_+$. Recall that we write $I_a := \overline{AaA}$ the ideal generated by a and $\operatorname{her} a := \overline{aAa}$ the hereditary subalgebra generated by a. Then a is obviously a full element in I_a and $\operatorname{her} a$ is a full hereditary subalgebra of I_a. Since A is separable, then so is I_a. Thus we can find a strictly positive element of I_a, that we write s_a. In fact, one can take $s_a := \sum_{n=1}^{\infty} a^{1/n}/2^n$.

Since $a \in \operatorname{her} s_a$, we know that $a \lesssim_{\mathrm{Cu}} s_a$. Observe that the canonical inclusion $i : \operatorname{her} a \hookrightarrow \operatorname{her} s_a = I_a$ is one of our standard morphisms (see Remark 2.1.5). That is, in the notation of Remark 2.1.5, $\chi_{as_a} = \mathrm{K}_1(i)$. Furthermore, by Corollary 3.1.4, we know that $i \otimes 1_{M(\mathcal{K})} : \operatorname{her} a \otimes \mathcal{K} \simeq \operatorname{her} s_a \otimes \mathcal{K}$ is a *-isomorphism. Hence, by functoriality of K_1, we get that $\chi_{as_a} : \mathrm{K}_1(\operatorname{her} a) \simeq \mathrm{K}_1(I_a)$ is an abelian group isomorphism. Notice that for any unitary element u of $\operatorname{her} a^\sim$, we have $\chi_{as_a}([u]_{\mathrm{K}_1(\operatorname{her} a)}) = [u]_{\mathrm{K}_1(I_a)}$.

Lemma 3.1.6. *Let* $A \in C^*$ *and let* $a, b \in (A \otimes \mathcal{K})_+$ *be such that* $a \lesssim_{\mathrm{Cu}} b$*. Let* s_a, s_b *be strictly positive elements of the ideals* I_a, I_b *respectively. Then the following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{U}(\operatorname{her} a^\sim) & \longrightarrow & \mathrm{K}_1(\operatorname{her} a) & \xrightarrow[\chi_{as_a}]{\simeq} & \mathrm{K}_1(I_a) \\ \downarrow{\scriptstyle \theta^\sim_{ab}} & & \downarrow{\scriptstyle \chi_{ab}} & & \downarrow{\scriptstyle \chi_{s_a s_b}} \\ \mathcal{U}(\operatorname{her} b^\sim) & \longrightarrow & \mathrm{K}_1(\operatorname{her} b) & \xrightarrow[\simeq]{\chi_{bs_b}} & \mathrm{K}_1(I_b) \end{array}$$

In particular, for any other strictly positive element $s_{a'}$ *of* I_a*, we have* $\operatorname{her} s_a = \operatorname{her} s_{a'}$ *and hence* $\chi_{s_a s_{a'}} = id_{\mathrm{K}_1(I_a)}$*, which finally gives us* $\chi_{as_a} = \chi_{as_{a'}}$*.*

Proof. By definition, $\chi_{ab} := \mathrm{K}_1(\theta^\sim_{ab})$ and hence the left-square is commutative. Furthermore, by transitivity of $\precsim_1$ (see Lemma 2.1.4), we know that $\chi_{s_a s_b} \circ \chi_{a s_a} = \chi_{a s_b} = \chi_{b s_b} \circ \chi_{ab}$. That is, the right square is commutative, which ends the proof. □

Definition 3.1.7. Let $A \in C^*$. Let $a \in (A \otimes \mathcal{K})_+$ and let s_A be any strictly positive element of I_a. Define $\delta_a := \chi_{a s_a}$. By Lemma 3.1.6, this does not depend on the strictly positive element s_a chosen. Moreover, $\delta_a : \mathrm{K}_1(\mathrm{her}\, a) \simeq \mathrm{K}_1(I_a)$ is a well-defined group isomorphism.
Now, without loss of generality, suppose that A is stable. Let $I, J \in \mathrm{Lat}(A)$ be ideals of A and let s_I, s_J be any strictly positive elements of I, J respectively. Suppose that $I \subseteq J$ or, equivalently $[s_I] \leq [s_J]$ in $\mathrm{Cu}(A)$. Define $\delta_{IJ} := \chi_{s_I s_J}$, which is also a well-defined group morphism (that is, it does not depend on the strictly positive elements chosen). In fact, from what we proved in Lemma 3.1.6, we have $\delta_{IJ} = \mathrm{K}_1(i)$, where $i : I \hookrightarrow J$ is the canonical inclusion. Thus, $\delta_{II} = id_{\mathrm{K}_1(I)}$.

Corollary 3.1.8. *Let $A \in C^*$ and let $a, b \in (A \otimes \mathcal{K})_+$ such that $[a] \leq [b]$ in $\mathrm{Cu}(A)$. Let u, v be unitary elements of* $\mathrm{her}\, a^\sim, \mathrm{her}\, b^\sim$ *respectively. We write $[u] := [u]_{\mathrm{K}_1(\mathrm{her}\, a)}$ and $[v] := [v]_{\mathrm{K}_1(\mathrm{her}\, b)}$.*
Then the following are equivalent:
(i) $\theta^\sim_{ab}(u) \sim_h v$ in $\mathrm{her}\, b^\sim$.
(ii) $\chi_{ab}([u]) = [v]$ in $\mathrm{K}_1(\mathrm{her}\, b)$.
(iii) $\delta_{I_a I_b}(\delta_a([u])) = \delta_b([v])$ in $\mathrm{K}_1(I_b)$, that is, $\delta_{I_a I_b}([u]_{\mathrm{K}_1(I_a)}) = [v]_{\mathrm{K}_1(I_b)}$.

Proof. Since $\mathrm{K}_1(\theta^\sim_{ab}) = \chi_{ab}$, we trivially obtain (i) is equivalent to (ii).
Furthermore, by the right-square of the commutative diagram in Lemma 3.1.6, we know that $\delta_{I_a I_b} \circ \delta_a([u]) = \delta_b \circ \chi_{ab}([u])$. And since δ_b is an isomorphism, we obtain that (ii) is equivalent to (iii). □

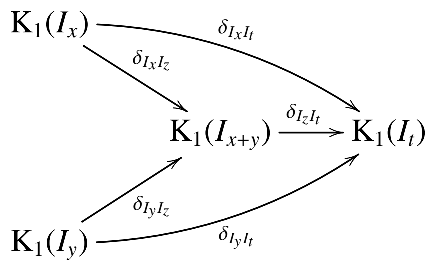
3.1.9. From Corollary 3.1.8, we conclude that whenever $[a] \leq [b]$ in $\mathrm{Cu}(A)$, the group morphism $\delta_{I_a I_b}$ defined in Definition 3.1.7 can replace K_1 of the standard morphism. That is, for any unitary elements u, v in $\mathrm{her}\, a^\sim, \mathrm{her}\, b^\sim$ respectively, we have $[(a, u)] \leq [(b, v)]$ in $\mathrm{Cu}_1(A)$ if and only if $[a] \leq [b]$ in $\mathrm{Cu}(A)$ and $\delta_{I_a I_b}([u]_{\mathrm{K}_1(I_a)}) = [v]_{\mathrm{K}_1(I_b)}$ in $\mathrm{K}_1(I_b)$.

Definition 3.1.10. Let $A \in C^*$ and let $I \in \mathrm{Lat}(A)$ be an ideal of A. We recall that $\mathrm{Cu}(I)$ is an ideal of $\mathrm{Cu}(A)$. We also recall that for $x \in \mathrm{Cu}(A)$, we write $I_x := \{y \in \mathrm{Cu}(A) \text{ such that } y \leq \infty.x\}$ the ideal of $\mathrm{Cu}(A)$ generated by x.
Define $\mathrm{Cu}_f(I) := \{[a] \in \mathrm{Cu}(A) \mid I_a = I\}$. Equivalently, $\mathrm{Cu}_f(I) := \{x \in \mathrm{Cu}(A) \mid I_x = \mathrm{Cu}(I)\}$. In other words, $\mathrm{Cu}_f(I)$ consists of the elements of $\mathrm{Cu}(A)$ that are full in $\mathrm{Cu}(I)$.

Remark 3.1.11. By Paragraph 1.3.10, we know that $\mathrm{Lat}(A) \simeq \mathrm{Lat}(\mathrm{Cu}(A))$ by sending any $I \in \mathrm{Lat}(A)$ to $\mathrm{Cu}(I)$. Furthermore, A is separable, hence we know that any ideal $I \in \mathrm{Lat}(A)$

and its image $\mathrm{Cu}(I) \in \mathrm{Lat}(\mathrm{Cu}(A))$ are singly-generated by a full element. In fact, a is a full element in I, (that is, $I_a = I$) if and only if $[a]$ is a full element in $\mathrm{Cu}(I)$. And in this case, we have $\mathrm{Cu}(I_a) = I_{[a]}$.
Hence, for notational purposes, we will indistinguishably use I_a or $I_{[a]}$, referring to one or the other. For instance, we might consider objects such as $\delta_{I_x I_y}$ or $\mathrm{K}_1(I_x)$, where $x, y \in \mathrm{Cu}(A)$, when we really mean $\delta_{I_a I_b}$ or $\mathrm{K}_1(I_a)$, where $a, b \in (A \otimes \mathcal{K})_+$ are representatives of x, y respectively.

Proposition 3.1.12. *Let $A \in C^*$. Then:*
(i) For every x in $\mathrm{Cu}(A)$ *and $k \in \overline{\mathbb{N}}$, $I_{k.x} = I_x$.*
(ii) For every $x \leq y$ in $\mathrm{Cu}(A)$*, we have the following: $I_{x+y} = I_y$, that is a priori different from I_x.*
(iii) Whenever x is not comparable with y in $\mathrm{Cu}(A)$*, then either $I_x = I_y = I_{x+y}$ or else I_{x+y}, I_y and I_x need not be the same. In this case, for every $t \geq z := x + y$, the following diagram is commutative:*



Proof. (i) By definition of I_x, this is trivial since $k(\infty.x) = \infty.x$.
(ii) If $x \leq y$ then $x \in I_y$, hence $I_{(x+y)} = I_y$.
(iii) If $I_x = I_y$ then $I_{(x+y)} = I_y$. Else, the result follows from transitivity of $\lesssim_1$ (see Lemma 2.1.4). □

3.1.13. We will now use all the above to get a new picture of the Cu$_1$-semigroup and its elements. Let us first state this lemma:

Lemma 3.1.14. *Let S be a* Cu$^\sim$*-semigroup and let T be a* PoM$^\sim$*. Let $f : S \longrightarrow T$ be a* PoM$^\sim$*-isomorphism. Then, T is a* Cu$^\sim$*-semigroup and f is a* Cu$^\sim$*-isomorphism. A fortiori, $S \simeq T$ as* Cu$^\sim$*-semigroups.*

Proof. Let $(t_k)_k$ be an increasing sequence in T. Since f is a surjective order-embedding, we can find an increasing sequence $(s_k)_k$ in S such that $f(s_k) = t_k$ for all k. We easily deduce that $f(\sup\limits_k s_k) \geq t_k$ for any $k \in \mathbb{N}$. Now, if $t \geq t_k$ for all $k \in \mathbb{N}$, then since there exists $s \in S$ such that $f(s) = t$ and f is an order-embedding, we have that $s \geq s_k$ for any k and thus

$t = f(s) \geq f(\sup_k s_k)$. Thus T satisfies (O1) and moreover f preserves suprema of increasing sequences.
Now let $x, y \in S$ be such that $x \ll y$. Let $(t_k)_k$ be an increasing sequence in T such that $f(y) \leq \sup_k t_k$. Since f is a surjective order-embedding, we know that there exists an increasing sequence $(s_k)_k$ in S such that $f(s_k) = t_k$ for any $k \in \mathbb{N}$. Let $s := \sup_k s_k$. Since $s \geq s_k$ for any k, then $f(s) \geq f(s_k) = t_k$ and passing to suprema, we deduce that $f(s) \geq f(y)$. Again, f is an order-embedding, so we deduce that $s \geq y$ in S. Now, since $x \ll y$, there exists $n \in \mathbb{N}$ such that $x \leq s_n$, which implies $f(x) \leq f(s_n) = t_n$. We conclude that $f(x) \ll f(y)$. From this, (O2) follows easily and hence f preserves the compact-containment relation. Axioms (O3) and (O4) are routine as well as the final conclusion. □

Definition 3.1.15. Let $A \in C^*$. Let us consider $S := \bigsqcup_{I \in \mathrm{Lat}(A)} \mathrm{Cu}_f(I) \times \mathrm{K}_1(I)$. We equip S with addition and order as follows: Let $(x, k) \in \mathrm{Cu}_f(I_x) \times \mathrm{K}_1(I_x)$ and $(y, l) \in \mathrm{Cu}_f(I_y) \times \mathrm{K}_1(I_y)$, (see Remark 3.1.11). Write $z := x + y$. Now define $(x, k) + (y, l) := (z, \delta_{I_x I_z}(k) + \delta_{I_y I_z}(l))$. Also, we write $(x, k) \leq (y, l)$ if: $x \leq y$ in $\mathrm{Cu}(A)$ and $\delta_{I_x I_y}(k) = l$.

Proposition 3.1.16. *Let* $A \in C^*$ *and let* $(S, +, \leq)$ *be the object defined in Definition 3.1.15. Then:*
(i) $(S, +, \leq)$ *is a* $\mathrm{Cu}^\sim$*-semigroup.*
(ii) The following map is a $\mathrm{Cu}^\sim$*-isomorphism:*

$$\begin{aligned} \xi : \mathrm{Cu}_1(A) &\longrightarrow S \\ [(a, u)] &\longmapsto ([a], \delta_a([u])) \end{aligned}$$

where $[a] := [a]_{\mathrm{Cu}(A)}$ *and* $[u] := [u]_{\mathrm{K}_1(\mathrm{her}\, a)}$.
Thus, whenever convenient, and many times in the future, we will describe elements of $\mathrm{Cu}_1(A)$ *as a couple* (x, k) *where* $x \in \mathrm{Cu}(A)$ *and* $k \in \mathrm{K}_1(I_x)$*; see Remark 3.1.11.*

Proof. By Definition 3.1.7 and Definition 3.1.10, the map $\mathrm{Cu}_1(A) \longrightarrow \bigsqcup_{I \in \mathrm{Lat}(A)} \mathrm{Cu}_f(I) \times \mathrm{K}_1(I)$ is well-defined. Further, by Proposition 3.1.12 addition and order are well-defined in S. Now let $a \in (A \otimes \mathcal{K})_+$. Since A has stable rank one, then so has $\mathrm{her}\, a$. Hence, by Proposition 1.1.17, we know that any element of $\mathrm{K}_1(\mathrm{her}\, a)$ lifts to a unitary in $\mathrm{her}\, a^\sim$ and that any two of those lifts are homotopic. Also δ_a is an isomorphism and obviously any two representatives of x in $(A \otimes \mathcal{K})_+$ are Cuntz equivalent. Thus for any $(x, k) \in \mathrm{Cu}(A) \times \mathrm{K}_1(I_x)$, there exist $a \in (A \otimes \mathcal{K})_+$ and $u \in \mathcal{U}(\mathrm{her}\, a^\sim)$ such that $[a] = x$ and $\delta_a[u] = k$. Moreover for any other lift (a', u'), by construction, gives us $[(a', u')] = [(a, u)]$. So we conclude that ξ is a set bijection.

Now, using Corollary 3.1.8 and Paragraph 3.1.9, we know that $[(a,u)] \leq [(b,v)]$ if and only if $\xi([(a,u)]) \leq \xi([(b,v)])$. Moreover, using Lemma 3.1.6, we have $\xi([(a,u)] + [(b,v)]) = \xi([(a,u)]) + \xi([(b,v)])$. In the end, we have ξ is a PoM$^\sim$-isomorphism. We finally conclude that S is a Cu$^\sim$-semigroup and that ξ is a Cu$^\sim$-isomorphism using Lemma 3.1.14. □

Remark 3.1.17. In this new picture, the positive elements of $\mathrm{Cu}_1(A)$ can be identified with $\{(x,0), x \in \mathrm{Cu}(A)\}$ (see Lemma 2.1.21). In other words, $\mathrm{Cu}_1(A)_+ \simeq \mathrm{Cu}(A)$ as Cu-semigroups.

3.1.18. We will end this section by describing morphisms from $\mathrm{Cu}_1(A)$ to $\mathrm{Cu}_1(B)$ in this new viewpoint of our invariant.

Lemma 3.1.19. *Let $A, B \in C^*$. Let I be an ideal of A and let $\phi : A \longrightarrow B$ be a *-homomorphism. Write $J := \overline{B\phi(I)B}$ the smallest ideal of B containing $\phi(I)$. Also write $\alpha := \mathrm{Cu}_1(\phi)$ and $\alpha_0 := \mathrm{Cu}(\phi)$.*
(i) For any $x \in \mathrm{Cu}_f(I)$, we have $\alpha_0(x) \in \mathrm{Cu}_f(J)$. That is, $I_{\alpha_0(x)} = \mathrm{Cu}(J)$ is the smallest ideal of $\mathrm{Cu}(B)$ containing $\alpha_0(\mathrm{Cu}(I))$.
(ii) Let $\xi_A : \mathrm{Cu}_1(A) \xrightarrow{\simeq} \bigsqcup_{I \in \mathrm{Lat}(A)} \mathrm{Cu}_f(I) \times \mathrm{K}_1(I)$ be the Cu$^\sim$-isomorphism in Proposition 3.1.16 (respectively ξ_B for B). For any $I \in \mathrm{Lat}(A)$, we define $\alpha_I := \mathrm{K}_1(\phi_{|I})$, where $\phi_{|I} : I \xrightarrow{\phi} J$. Then for any (x,k) with $x \in \mathrm{Cu}_f(I)$ and $k \in \mathrm{K}_1(I)$ (see Proposition 3.1.16), we have $\alpha(\xi^{-1}(x,k)) = (\alpha_0(x), \alpha_I(k))$.
Thus, we may abuse the language and describe morphisms $\alpha := \mathrm{Cu}_1(\phi)$ from $\mathrm{Cu}_1(A)$ to $\mathrm{Cu}_1(B)$, whenever convenient, as couples $\alpha := (\alpha_0, \{\alpha_I\}_{I \in \mathrm{Lat}(A)})$, where $\alpha_0 := \mathrm{Cu}(\phi)$ and $\alpha_I := \mathrm{K}_1(\phi_{|I})$.

Proof. By functoriality of Cu and Paragraph 1.3.10, we know that $\mathrm{Cu}(J)$ is the smallest ideal of $\mathrm{Cu}(B)$ that contains $\alpha_0(\mathrm{Cu}(I))$. Now let $x \in \mathrm{Cu}_f(I)$. Then $\alpha_0(x) \in \alpha_0(\mathrm{Cu}(I))$. Hence $I_{\alpha_0(x)} \subseteq \mathrm{Cu}(J)$. However, since x is full in $\mathrm{Cu}(I)$, we have $\alpha_0(\mathrm{Cu}(I)) \subseteq I_{\alpha_0(x)}$. By minimality of $\mathrm{Cu}(J)$ we deduce that $I_{\alpha_0(x)} = \mathrm{Cu}(J)$, that is, $\alpha_0(x) \in \mathrm{Cu}_f(J)$, which proves (i).
(ii) Let (x,k) be an element of $\mathrm{Cu}_1(A)$, where $x \in \mathrm{Cu}(A)$ and $k \in \mathrm{K}_1(I_x)$. Let (a,u) be a representative of (x,k), that is, $\xi([(a,u)]) = (x,k)$ as in Proposition 3.1.16. That is, $[a] = x$ in $\mathrm{Cu}(A)$ and $\delta_a([u]_{\mathrm{K}_1(\mathrm{her}\, a)}) = [u]_{\mathrm{K}_1(I_a)} = k$. We know that

$$\begin{aligned}
\alpha(\xi^{-1}(x,k)) &= \alpha([a,u]) \\
&= [(\phi(a)), \phi^\sim(u))] \\
&= ([\phi(a)]_{\mathrm{Cu}(B)}, \delta_{\phi(a)}([\phi^\sim(u)]_{\mathrm{K}_1(\mathrm{her}\,\phi(a)^\sim)})) \\
&= ([\phi(a)]_{\mathrm{Cu}(B)}, [\phi^\sim(u)]_{\mathrm{K}_1(I^\sim_{\phi(a)})})
\end{aligned}$$

Hence $\alpha(\xi^{-1}(x,k)) = (\alpha_0(x), \alpha_I(k))$ as desired. □

3.2 Ideal structure in Cu$^\sim$

3.2.1. In this section, we define ideals in the category Cu$^\sim$. The idea is to work with semigroups that are countably-based, using the same definition as for Cu-semigroups. That is, we say that a Cu$^\sim$-semigroup S is countably-based if there exists a countable subset $B \subseteq S$ such that for any pair $a' \ll a$, there exists $b \in B$ such that $a' \leq b \ll a$. We will also use concepts from Domain Theory (see [32], [44]).

Definition 3.2.2. [32, Definition II.1.3] Let S be a Cu$^\sim$-semigroup. A subset $O \subseteq S$ is called Scott-open if:
(i) O is an upper set, that is, for any $y \in S$, $y \geq x \in O$ implies $y \in O$.
(ii) For any $x \in O$, there exists $x' \ll x$ such that $x' \in O$. Equivalently, for any increasing sequence of S whose supremum belongs to O, there exists an element of the sequence also in O.
Dually we say that $F \subseteq S$ is Scott-closed if $S \setminus F$ is Scott-open, that is, if it is a lower set and closed under suprema of increasing sequences.

Remark 3.2.3. Let us check the equivalence in (ii) in the above definition: Let O be an upper set of S and let $x \in O$. Suppose there exists $x' \ll x$ such that $x' \in O$. Let $(x_n)_n$ be any increasing sequence whose supremum is x. By definition of $\ll$, there exists $x_n \geq x'$, hence x_n is also in O.
Conversely, using (O2), there exists a $\ll$-increasing sequence $(x_n)_n$ whose supremum is x. By hypothesis, there exists n such that $x_n \in O$, and by construction $x_n \ll x$. This finishes the proof.

Definition 3.2.4. Let S be a Cu$^\sim$-semigroup. We define the following axioms:
(PD): We say that S is *positively directed* if, for any $x \in S$, there exists $p_x \in S$ such that $x + p_x \geq 0$.
(PC): We say that S is *positively convex* if, for any $x, y \in S$ such that $y \geq 0$ and $x \leq y$, we have $x + y \geq 0$.

Lemma 3.2.5. *Let $A \in C^*$. Then* $\mathrm{Cu}_1(A)$ *is positively directed and positively convex.*

Proof. Let $A \in C^*$. Using the picture of Proposition 3.1.16 consider $(x,k) \in \mathrm{Cu}_1(A)$, where $x \in \mathrm{Cu}(A)$ and $k \in \mathrm{K}_1(I_x)$, we deduce that $(x,k) + (x,-k) = (2x, 0) \geq 0$, and so $\mathrm{Cu}_1(A)$ is positively directed. Now let $(y, 0)$ be a positive element in $\mathrm{Cu}_1(A)$ such that $(x, k) \leq (y, 0)$. Since $(x, k) \leq (y, 0)$, we know that $\delta_{I_x I_y}(k) = 0$. Therefore, $\delta_{I_x I_{x+y}}(k) = 0$, and we deduce that $(x, k) + (y, 0) = (x + y, 0)$ is a positive element in $\mathrm{Cu}_1(A)$, which finishes the proof. □

Remark 3.2.6. If S is positively convex, then the only negative element of S is 0.

Definition 3.2.7. Let S be a Cu~-semigroup. We define $S_{max} := \{x \in S \mid \text{ if } y \geq x, \text{ then } y = x\}$. This subset can be interpreted as the set of maximal elements of S.

Proposition 3.2.8. *Let S be a countably-based positively directed* Cu~*-semigroup. Then S_{max} is not empty and is an abelian group with neutral element $e_{S_{max}} := y + p_y$, where y is any element of S_{max} and p_y any element of S such that $y + p_y \geq 0$. In fact, the inverse of y is $2p_y + y$.*

Proof. By assumption, for any $x \in S$, there exists at least one element $p_x \in S$, such that $x + p_x \geq 0$. We will first show that S_{max} is closed under addition.
Let y, z be elements in S_{max} and let $x \in S$ be such that $x \geq y+z$. We first have $x+p_z \geq y+z+p_z \geq y$ and $x + p_y \geq z + y + p_y \geq z$, which gives us the following equalities: $x + p_z = y + z + p_z = y$ and $x + p_y = z + y + p_y = z$. Obviously $x \leq x + p_z + z = x + p_z + x + p_y = y + z$ and since $x \geq y + z$, we have $x = y + z$ which tells us that S_{max} is closed under addition.
Now, let us show the following: for any $z \in S_{max}$ and any $p_z \in S$ such that $z + p_z \geq 0$, we have $z + p_z \in S_{max}$. Let $x \in S$ be such that $x \geq z + p_z$. We know that for any $y \in S_{max}$, $y + z + p_z = y$. In particular, $2z + p_z = z$. Also, $x + z \geq 2z + p_z = z$. Hence $x + z = z$. Finally compute that $x \leq x + z + p_z = z + p_z$. Therefore $x = z + p_z$, that is, $z + p_z \in S_{max}$.
Next for any y, z elements of S_{max}, we have $y + p_y + z + p_z \geq z + p_z, y + p_y$, which by what we have just proved gives us $y + p_y = y + p_y + z + p_z = z + p_z$. Hence, the positive element $e_{S_{max}} := y + p_y$ belongs to S_{max} and is independent of y and p_y. If $z \in S_{max}$, since $e_{S_{max}} \geq 0$, $z + e_{max} \geq z$ and we obtain $z + e_{S_{max}} = z$. Thus we have that S_{max} is an abelian monoid with well-defined neutral element $e_{S_{max}}$.
We already know that $z + (2p_z + z) = e_{S_{max}}$ for any $z \in S_{max}$. Let us show that $2p_z + z$ belongs to S_{max} for any $z \in S_{max}$ and any $p_z \in S$. Let $x \geq 2p_z + z$. Then $x + z \geq e_{S_{max}}$, hence $x + z = e_{S_{max}}$. On the other hand, $x \leq x + z + p_z = e_{S_{max}} + p_z = 2p_z + z$. Therefore $2p_z + z$ belongs to S_{max} and is the (unique) inverse of z, which finishes the proof that S_{max} is an abelian group.
Lastly, observe that $\nu_+(S)$ (see Lemma 2.1.21) is a countably-based Cu-semigroup. Therefore it has a maximal element which ensure us the existence of a maximal positive element in S and a fortiori that S_{max} is a non-empty abelian group. □

Remark 3.2.9. In the context of Proposition 3.2.8, $e_{S_{max}}$ is the only positive element of S_{max}, and the only positive maximal element of S. We will see later that whenever A is separable, $\mathrm{Cu}_1(A)_{max} \simeq \mathrm{K}_1(A)$ with neutral element $(\infty_{\mathrm{Cu}(A)}, 0_{\mathrm{K}_1(A)})$.

Lemma 3.2.10. *Let S be a countably-based* Cu~*-semigroup. Then the following are equivalent:*
(i) S is positively directed.
(ii) For any $x \in S$, there exists a unique $p_x \in S_{max}$ such that $x + p_x \geq 0$.
(iii) S_{max} is an absorbing abelian group in S whose neutral element $e_{S_{\max}}$ is positive.

Proof. (ii) implies (i) is clear.
Let us show that (i) implies (iii): We know from Proposition 3.2.8 that S_{max} is an abelian group whose neutral element is $e_{S_{max}} \geq 0$. Let $x \in S$ and let $p \in S_{max}$, we know there exists $y \in S$ such that $x + y \geq 0$. Hence $x + y + p \geq p$. Let $z \in S$ be such that $z \geq x + p$. we have $z+y \geq x+y+p = p$ and hence $z+y = p$. Now since $x+y \geq 0$, we have $z \geq x+p = x+z+y \geq z$ which gives us $z = x + p$, that is, $x + p \in S_{max}$ for any $x \in S$ and $p \in S_{max}$. This shows that S_{max} is an absorbing abelian group in S.
Let us show now that (iii) implies (ii): Let $x \in S$ and write $e := e_{S_{max}}$. Let $q := x + e$. Note that q belongs to S_{max} by (iii). Denote by p_x the inverse of q in S_{max}. we have $x + e + p_x = e$, and $x + p_x \in S_{max}$ by assumption. Therefore $x + p_x + e = x + p_x = e \geq 0$. Now suppose there exists another $r \in S_{max}$ such that $r + x \geq 0$. Then $r + x + p_x = p_x$. However $x + p_x = e$, hence $r = p_x$, which ends the proof. □

Remark 3.2.11. One can notice all the proofs above hold in a positively directed and positively convex partially ordered monoid S, but one cannot know for sure that S_{max} is not empty. Indeed, it suffices to assume axiom (O1) together with the countably-based property to ensure that S_{max} contains at least one element (the maximal positive element of S).
In fact, for a Cu-semigroup S, we have that S_{max} is either empty, or the trivial group consisting of the largest element of S.

Definition 3.2.12. Let S be a positively directed Cu~-semigroup and let $x \in S$. We define $P_x := \{y \in S, x + y \geq 0\}$.

Remark 3.2.13. It is easy clear that $P_x \neq \emptyset$. In fact, P_x (and also $x + P_x$) is a Scott-open set in S. In particular, $S_+ = P_0$ is Scott-open in S.
Also, if $x \in S$, then P_x is obviously an upper set. Let $y \in P_x$. Using that $0 \ll 0$ and (O2), we can construct a $\ll$-increasing sequence towards y, and we have by construction some $y_n \ll y$ such that $x + y_n \geq 0$.

Definition 3.2.14. Let S be countably-based positively directed Cu˜-semigroup. Let M be a subset of S. We say M is *positively stable* if M satisfies axiom (PD) and moreover, for any $x \in S$, $(x + P_x) \bigcap M \neq \emptyset$ implies that $x \in M$.

Definition 3.2.15. Let S be countably-based positively directed and positively convex Cu˜-semigroup. We say that $I \subseteq S$ is an order-ideal (or ideal) of S if I is a Scott-closed, positively stable submonoid of S.
In this case, I is also a countably-based positively directed and positively convex Cu˜-semigroup, and it order-embeds canonically into S (that is, the inclusion map is Scott-continuous).
The set of ideals of S will be denoted $\mathrm{Lat}(S)$.
Lastly, we say that an ideal I of S is *simple* if it only contains the trivial ideal $\{0\}$ and I.

3.2.16. We naturally want to define the ideal generated by an element. However, we cannot ensure that the intersection of ideals is still an ideal. In fact, being positively directed is not preserved under intersection, so we will define the ideal generated by an element abstractly as follows:

Definition 3.2.17. Given $x \in S$, we define $\mathrm{Idl}(x)$ as the smallest ideal of S containing x, that is, $x \in \mathrm{Idl}(x)$ and for any J ideal of S containing x we have $J \supseteq \mathrm{Idl}(x)$. Note that this ideal might not exist.

3.2.18. Here we offer an example of two ideals of a countably-based positively directed and positively convex Cu˜-semigroup, whose intersection fails to be positively directed, and hence fails to be an ideal:
Let S be the subset of $\overline{\mathbb{N}}^3 \times \mathbb{Z}$ defined as follows:

$$S := \{((n_1, n_2, n_3), k) \in \overline{\mathbb{N}}^3 \times \mathbb{Z} \mid k \geq 0, \text{ if } n_1 = n_3 = 0, \text{ and } k = 0, \text{ if } n_1 = n_2 = n_3 = 0\}.$$

We put on this set a component-wise sum and we define for any two pairs: $(g, k) \leq (h, l)$ if $g \leq h$ in $\overline{\mathbb{N}}^3$ and $k = l$ in $\mathbb{Z}$. Notice that $S_+ = \overline{\mathbb{N}}^3 \times \{0\}$. One can check that $(S, +, \leq)$ is a countably-based positively directed and positively convex Cu˜-semigroup.
Now consider $I_1 := ((\overline{\mathbb{N}}\times\overline{\mathbb{N}}\times\{0\})\times\mathbb{Z})\cap S$ and $I_2 := ((\{0\}\times\overline{\mathbb{N}}\times\overline{\mathbb{N}})\times\mathbb{Z})\cap S$. Again, one can check that those are ideals of S as defined earlier. However, $I_1 \cap I_2 = ((\{0\} \times \overline{\mathbb{N}}_* \times \{0\}) \times \mathbb{Z}_+) \sqcup \{0_S\}$ is not positively directed. Indeed, let $x := ((0, n, 0), 1) \in I_1 \cap I_2$. Observe that any element $y \in I_1 \cap I_2$ is of the form $((0, n, 0), k)$ for some $n \in \mathbb{N}$ and $k \geq 0$. Thus, there is no $y \in I_1 \cap I_2$ such that $x + y \geq 0$ and hence $I_1 \cap I_2$ is not positively directed.

Proposition 3.2.19. *Let S be countably-based positively directed and positively convex* Cu$^\sim$*-semigroup. Let x be a positive element of S. Define $I_x := \{y \in S \mid$ there is $y' \in S$ with $0 \leq y + y' \leq \infty.x\}$. Then $I_x = Idl(x)$.*

Proof. First, we show that I_x is a submonoid of S that contains x. Using (O1), we know that $\infty.x := \sup_{n\in\mathbb{N}} n.x$ is a positive element. Moreover $0 \leq 0 + 0 \leq \infty.x$, hence $0 \in I_x$. We also know that for any n, m in $\overline{\mathbb{N}}$, $0 \leq n.x + m.x \leq \infty.x$. So we get that $\{n.x\}_{n\in\overline{\mathbb{N}}} \subseteq I_x$. Let y_1, y_2 in I_x. Then one easily checks that $0 \leq (y_1 + y_2) + (y_1' + y_2') \leq 2.(\infty.x) = \infty.x$, hence I_x is closed under addition. This proves it is a submonoid of S that contains x.
Furthermore, we claim that $\infty.x$ is the maximal positive element of I_x: let $y \in I_x$ such that $y \geq 0$. There exists $y' \in I_x$ such that $0 \leq y + y' \leq \infty.x$. Since $y \geq 0$, we get that $y' \leq y + y' \leq \infty.x$. So, by axiom (PC) we deduce that $0 \leq y' + \infty.x$. Now we add y on both sides to get that $y \leq y + y' + \infty.x \leq 2.(\infty.x) = \infty.x$, which proves the claim.

We will now prove that I_x is closed under suprema of increasing sequences. Let $(y_n)_n$ be an increasing sequence in I_x. Let y_0' be such that $0 \leq y_0 + y_0' \leq \infty.x$, where y_0 is the first term of $(y_n)_n$. Observe that y_0' belongs to I_x. Since I_x is closed under addition, for any $n \in \mathbb{N}$, we have $y_n + y_0' \in I_x$. Therefore we can choose $z_n \in I_x$ such that $(0 \leq)\, y_n + y_0' + z_n \leq \infty.x$. Finally choose $z_n' \in I_x$ such that $0 \leq z_n + z_n' \leq \infty.x$.
Thus, we have on the one hand that $0 \leq y_n + y_0' \leq (y_n + y_0') + (z_n + z_n')$ and on the other hand that $(y_n + y_0' + z_n) + z_n' \leq \infty.x + z_n'$ for any $n \in \mathbb{N}$. Now since I_x is submonoid of S that contains x and $z_n' \in I_x$, we get that $\infty.x + z_n'$ is a positive element of I_x. Now since $\infty.x = 2.(\infty.x)$, we have $(\infty.x + z_n') = 2.(\infty.x) + z_n' \geq \infty.x$. By maximality of $\infty.x$ in I_x, we get that $0 \leq y_n + y_0' \leq \infty.x$, for any $n \in \mathbb{N}$. Using Cuntz axioms (O3)-(O4), we pass to suprema and we obtain $0 \leq y + y_0' \leq \infty.x$, that is, $y \in I_x$. So I_x is closed under suprema of increasing sequences.

We also know that I_x is positively directed. Thus, by Proposition 3.2.8, we know that $(I_x)_{max}$ is a (non empty) abelian group. Indeed, it contains at least $\infty.x$, its neutral element, which again is the unique maximal positive element of I_x.
Let us show that I_x is positively stable. Take any $z \in S$ such that there exists z' with $0 \leq z + z'$ and $(z + z') \in I_x$. We know there is a $y \in I_x$ such that $0 \leq z + z' + y \leq \infty.x$. Hence $z \in I_x$.

Next, we have to show that I_x is a lower set. Let $z \leq y$ with $y \in I_x$. We know that there exists $y' \in I_x$ such that $0 \leq y + y' \leq \infty.x$. Since $z + y' \leq y + y'$, we deduce by axiom (PC) that

$0 \leq z + y' + y + y' \leq 2(y + y') \leq \infty.x$. Therefore $z \in I_x$, that is, I_x is a lower set, which ends the proof that I_x is an ideal of S containing x.

Lastly, let J be an ideal of S containing x. Then it contains $\infty.x = e_{(I_x)_{max}}$. Thus if $y \in I_x$, we know that there exists $y' \in I_x$ such that $0 \leq y + y' \leq \infty.x$, and therefore $y + y' \in (y + P_y) \cap J$. Since J is positively stable, this implies that $y \in J$. We obtain $J \supseteq I_x$, which gives us that I_x is the ideal generated by x. □

Corollary 3.2.20. *Let S be countably-based positively directed and positively convex* Cu~*-semigroup, and let I be an ideal of S. Then I is singly-generated. In fact, $I = I_{e_{I_{max}}}$.*

Proof. For any $x \in I$, there exists by Proposition 3.2.8 a unique $p_x \in I$ such that $x + p_x = e_{I_{max}}$. Since $I_{e_{I_{max}}}$ is positively stable, we have $x \in I_{e_{I_{max}}}$. Conversely, if $x \in I_{e_{I_{max}}}$, then there exists $x' \in I_{e_{I_{max}}}$ such that $0 \leq x + x' \leq e_{I_{max}}$. Since I is positively stable, we obtain $x \in I$, which ends the proof. □

Corollary 3.2.21. *Let S be countably-based positively directed and positively convex* Cu~*-semigroup, and let I, J be two ideals of S. Then $e_{I_{max}} \leq e_{J_{max}}$ if and only if $I \subseteq J$.*

Proof. Suppose $e_{I_{max}} \leq e_{J_{max}}$. We easily see that $I_{e_{I_{max}}} \subseteq J_{e_{J_{max}}}$. By Corollary 3.2.20, we obtain $I \subseteq J$. The converse is trivial by maximality of $e_{J_{max}}$. □

Theorem 3.2.22. *Let S be countably-based positively directed and positively convex* Cu~*-semigroup. Let us consider the following map:*

$$\begin{aligned} \Phi : \mathrm{Lat}(S) &\longrightarrow \mathrm{Lat}(\nu_+(S)) \\ I &\longmapsto \nu_+(I) \end{aligned}$$

Then it is a well-defined ordered set isomorphism and the inverse map is defined as follows: For any Cu*-ideal J of $\nu_+(S)$, $\Phi^{-1}(J) := I_{e_{J_{max}}}$.*

Proof. We know that $\nu_+(S)$ is a countably-based Cu-semigroup, hence for any ideal $J \in \mathrm{Lat}(\nu_+(S))$, we have $J_{max} = \{e_{J_{max}}\}$. In fact, J is generated (as Cu ideal) by its maximal element $e_{J_{max}}$, that is, $J = \{x \in \nu_+(S) \mid x \leq e_{J_{max}}\}$. Now since $e_{\nu_+(I)_{max}} = e_{I_{max}}$, we deduce by Corollary 3.2.21 that Φ and Φ^{-1} are well-defined ordered set maps that are inverses of one another. □

Corollary 3.2.23. *Let S be countably-based positively directed and positively convex* Cu~*-semigroup. Then:*

(i) $\mathrm{Lat}(S)$ *is a complete lattice, with the structure inherited from* Φ^{-1} *of Theorem 3.2.22.*
(ii) For any $A \in C^*$*, we have* $\mathrm{Lat}(A) \simeq \mathrm{Lat}(\mathrm{Cu}_1(A))$ *as complete lattices.*
(iii) For any $A \in C^*$ *and any* $I \in \mathrm{Lat}(A)$*, we have* $\mathrm{Cu}_1(I) \in \mathrm{Lat}(\mathrm{Cu}_1(A))$*. In fact, any ideal* $J \in \mathrm{Lat}(\mathrm{Cu}_1(A))$ *is of the form* $\mathrm{Cu}_1(I)$*, for some* $I \in \mathrm{Lat}(A)$*, and* $\mathrm{Cu}_1(I)$ *is simple if and only if* I *is simple.*

Proof. (i) From Paragraph 1.3.10, we know that $\mathrm{Lat}(\nu_+(S))$ is a complete lattice, so the Set isomorphism Φ^{-1} takes the lattice structure onto $\mathrm{Lat}(S)$ to make it a complete lattice.
(ii)-(iii) Let A be a separable C^*-algebra. We also know that $\mathrm{Lat}(A) \simeq \mathrm{Lat}(\mathrm{Cu}(A))$ by sending any $I \in \mathrm{Lat}(A)$ to $\mathrm{Cu}(I)$. One can easily check that $\Phi^{-1}(\mathrm{Cu}(I)) \simeq \mathrm{Cu}_1(I)$, hence any (resp simple) ideal of $\mathrm{Cu}_1(A)$ is of the form $\mathrm{Cu}_1(I)$ for some (resp simple) $I \in \mathrm{Lat}(A)$. □

Remark 3.2.24. Let us explicitly compute the lattice structure on $\mathrm{Cu}_1(A)$ for any $A \in C^*$. Let $I, J \in \mathrm{Lat}(A)$, then $\mathrm{Cu}_1(I) \wedge \mathrm{Cu}_1(J) = \mathrm{Cu}_1(I \cap J)$ and $\mathrm{Cu}_1(I) \vee \mathrm{Cu}_1(J) = \mathrm{Cu}_1(I + J)$.

Lemma 3.2.25. *Let* S, T *be countably-based positively directed and positively convex* $\mathrm{Cu}^{\sim}$*-semigroups. Let* $\alpha : S \longrightarrow T$ *be a* $\mathrm{Cu}^{\sim}$*-morphism and let* I, I' *be two ideals of* S *such that* $I \subseteq I'$*. Then:*
(i) $J := I_{\alpha(e_{I_{max}})}$ *is the smallest ideal in* $\mathrm{Lat}(T)$ *that contains* $\alpha(I)$ *and* $J' := I_{\alpha(e_{I'_{max}})}$ *is the smallest ideal in* $\mathrm{Lat}(T)$ *that contains* $\alpha(I')$*. Also, we have that* $J \subset J'$*.*
(ii) Define $\alpha_{|I} : I \longrightarrow J$*, the restriction of* α *that has codomain* J*, respectively* $\alpha_{|I'}, I', J'$*. Then the following square is commutative:*

$$\begin{array}{ccc} I & \xrightarrow{\;i\;} & I' \\ {\scriptstyle\alpha_{|I}}\downarrow & & \downarrow{\scriptstyle\alpha_{|I'}} \\ J & \xrightarrow[\;i\;]{} & J' \end{array}$$

where i stands for the canonical inclusions.

Proof. Since α is order-preserving, $\alpha_{|J}$ and $\alpha_{|J'}$ are well-defined. Besides, we know that for any $y \in I$, there exists y', such that $0 \leq y + y' \leq e_{I_{max}}$, hence we have $0 \leq \alpha(y) + \alpha(y') \leq \infty.\alpha(e_{I_{max}})$. Therefore $\alpha(y) \in J$ and we obtain that $\alpha(I) \subseteq J$, respectively $\alpha(I') \subseteq J'$. Now, by Corollary 3.2.21, we have $e_{I_{max}} \leq e_{I'_{max}}$ and hence $\alpha(e_{I_{max}}) \leq \alpha(e_{I'_{max}})$. Thus $J \subseteq J'$ and we see the square is commutative. □

3.2.26. In the sequel, when we speak of the restriction of a $\mathrm{Cu}^{\sim}$-morphism to an ideal, we will always mean, unless stated otherwise, the map defined above. That is, we also restrict the

codomain. Using notations of Lemma 3.2.25, notice that $\alpha_{|I}(e_{I_{max}}) = e_{J_{max}}$.
As done with ν_c and ν_+ in the previous chapter (see Lemma 2.1.21, Proposition 2.4.4), we define a functor ν_{max} in order to obtain the abelian group S_{max} in a functorial way.

Proposition 3.2.27. *Let $\alpha : S \longrightarrow T$ be a* Cu~*-morphism between countably-based positively directed and positively convex* Cu~*-semigroups S, T. Let $\alpha_{max} := \alpha + e_{T_{max}}$. Then α_{max} is a* AbGp*-morphism from S_{max} to T_{max}.*
Thus we obtain a functor:

$$\begin{aligned} \nu_{max} : \mathrm{Cu}^{\sim} &\longrightarrow \mathrm{AbGp} \\ S &\longmapsto S_{max} \\ \alpha &\longmapsto \alpha_{max} \end{aligned}$$

Proof. Let us first show that α_{max} is a group morphism. For any $s \in S_{max}$, we know that $(\alpha(s) + e_{T_{max}}) \in T_{max}$. Now, since α is a Cu~-morphism, we have $\alpha_{max}(s_1) + \alpha_{max}(s_2) = \alpha(s_1) + \alpha(s_2) + 2e_{T_{max}} = \alpha(s_1 + s_2) + e_{T_{max}} = \alpha_{max}(s_1 + s_2)$, for any s_1, s_2 elements of S_{max}.
Let us now show that ν_{max} satisfies the functor properties. Trivially, $\nu_{max}(id) = id$. Let $\alpha : S \longrightarrow T$ and $\beta : T \longrightarrow R$ be two Cu~-morphisms. Let $s \in S_{max}$. Then:

$$\begin{aligned} \beta_{max} \circ \alpha_{max}(s) &= \beta(\alpha(s) + e_{T_{max}}) + e_{R_{max}} \\ &= (\beta \circ \alpha)_{max}(s) \end{aligned}$$

Hence $\nu_{max}(\beta \circ \alpha) = \nu_{max}(\beta) \circ \nu_{max}(\alpha)$. □

Definition 3.2.28. Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be covariant functors. Recall that a *natural transformation $\eta : F \Rightarrow G$* is a collection of maps $\eta_C : F(C) \longrightarrow G(C)$ defined in a natural way (that is, defined in the same way for every object C) such that for any morphism $h : C_1 \longrightarrow C_2$ in $\mathcal{C}$, the following square is commutative in $\mathcal{D}$:

$$\begin{array}{ccc} F(C_1) & \xrightarrow{\eta_{C_1}} & G(C_1) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ F(C_2) & \xrightarrow[\eta_{C_2}]{} & G(C_2) \end{array}$$

Moreover, if η_C is an isomorphism for some C (hence for any $C \in \mathcal{C}$), we say that there exists a *natural isomorphism* between F and G and we write $F \simeq G$.
Dually, one can define those notions for contravariant functors, by reversing the vertical arrows in the square.

3.2.29. In the next theorem, we use the picture of the Cu_1-semigroup described in Proposition 3.1.16.

Theorem 3.2.30. *Let $A \in C^*$. Then we have the following natural isomorphisms in* Cu *and* AbGp *respectively:*

$$\begin{array}{cc} \mathrm{Cu}_1(A)_+ \simeq \mathrm{Cu}(A) & \qquad \mathrm{Cu}_1(A)_{max} \simeq \mathrm{K}_1(A) \\ (x,0) \longmapsto x & \qquad (\infty_A, k) \longmapsto k \end{array}$$

In fact, we have the following natural isomorphisms between functors that are defined between $C^* \longrightarrow$ Cu *and $C^* \longrightarrow$* AbGp *respectively:*

$$\nu_+ \circ \mathrm{Cu}_1 \simeq \mathrm{Cu} \qquad\qquad \nu_{max} \circ \mathrm{Cu}_1 \simeq \mathrm{K}_1$$

Proof. From the construction of the order in $\mathrm{Cu}_1(A)$, we know that any positive element of $\mathrm{Cu}_1(A)$ is of the form $(x,0)$ for some $x \in \mathrm{Cu}(A)$. Set $\infty_A := [s_{A\otimes\mathcal{K}}] = \sup\limits_{n\in\mathbb{N}} n.[s_A]$, the largest element of $\mathrm{Cu}(A)$. We know that any maximal element of $\mathrm{Cu}_1(A)$, that is, any element of $\mathrm{Cu}_1(A)_{max}$, is of the form (∞_A, k) for some $k \in \mathrm{K}_1(A)$. Hence we easily get the two canonical isomorphisms of the statement.
Now consider a *-homomorphism $\phi : A \longrightarrow B$. Let $(x,0)$ be a positive element of $\mathrm{Cu}_1(A)$. By Lemma 3.1.19 we have $\mathrm{Cu}_1(\phi)_+(x,0) = (\mathrm{Cu}(\phi)(x), 0)$. Let (∞_A, k) be in $\mathrm{Cu}_1(A)_{max}$. Again by Lemma 3.1.19 we know that

$$\begin{aligned} \mathrm{Cu}_1(\phi)_{max}(\infty_A, k) &= (\mathrm{Cu}(\phi)(\infty_A), \mathrm{Cu}_1(\phi)_A(k)) + (\infty_B, 0) \\ &= (\infty_B, \delta_{I_{\phi(\infty_A)}B} \circ \mathrm{Cu}_1(\phi)_A(k)) \\ &= (\infty_B, \mathrm{K}_1(\phi)(k)). \end{aligned}$$

This exactly gives us that

$$\begin{array}{ccc} \mathrm{Cu}_1(A)_+ & \xrightarrow{\simeq} & \mathrm{Cu}(A) \\ {\scriptstyle \mathrm{Cu}_1(\phi)_+}\downarrow & & \downarrow{\scriptstyle \mathrm{Cu}(\phi)} \\ \mathrm{Cu}_1(B)_+ & \xrightarrow[\simeq]{} & \mathrm{Cu}(B) \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Cu}_1(A)_{max} & \xrightarrow{\simeq} & \mathrm{K}_1(A) \\ {\scriptstyle \mathrm{Cu}_1(\phi)_{max}}\downarrow & & \downarrow{\scriptstyle \mathrm{K}_1(\phi)} \\ \mathrm{Cu}_1(B)_{max} & \xrightarrow[\simeq]{} & \mathrm{K}_1(B) \end{array}$$

are commutative squares. □

Corollary 3.2.31. *Let $A, B \in C^*$. Let $I \in$* Lat(A) *be an ideal of A and let $\phi : A \longrightarrow B$ be a *-homomorphism. Write $\alpha := \mathrm{Cu}_1(\phi)$ and $J := \overline{B\phi(I)B}$. Let us use the same notations as in*

Lemma 3.1.19, that is, $\alpha = (\alpha_0, \{\alpha_I\}_{I\in \mathrm{Lat}(A)})$. Then:
(i) $\nu_+(\alpha_{|\mathrm{Cu}_1(I)}) = \alpha_{0|\mathrm{Cu}(I)}$ and $\nu_{max}(\alpha_{|I}) = \alpha_I$.
(ii) Let $I' \in \mathrm{Lat}(A)$ such that $I' \supseteq I$. Then the following squares are commutative in their respective categories:

$$\begin{array}{ccc} \mathrm{Cu}(I) & \xrightarrow{\ i\ } & \mathrm{Cu}(I') \\ {\scriptstyle \alpha_{0|\mathrm{Cu}(I)}}\downarrow & & \downarrow{\scriptstyle \alpha_{0|\mathrm{Cu}(I')}} \\ \mathrm{Cu}(J) & \xrightarrow[\ i\ ]{} & \mathrm{Cu}(J') \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{K}_1(I) & \xrightarrow{\ \delta_{II'}\ } & \mathrm{K}_1(I') \\ {\scriptstyle \alpha_I}\downarrow & & \downarrow{\scriptstyle \alpha_{I'}} \\ \mathrm{K}_1(J) & \xrightarrow[\ \delta_{JJ'}\ ]{} & \mathrm{K}_1(J') \end{array}$$

where the maps i stand for the natural inclusions in Cu*.*
Observe that (ii) follows trivially from functoriality of Cu *and* K$_1$*, but we illustrate here how it can also be derived from our methods.*

Proof. (i) Using the isomorphisms of lattices of Theorem 3.2.22, we have $\mathrm{Cu}_1(J)$ is the smallest ideal of $\mathrm{Cu}_1(B)$ that contains $\alpha(\mathrm{Cu}_1(I))$. Hence, $\alpha_{|\mathrm{Cu}_1(I)}$ defined in Lemma 3.2.25 has codomain $\mathrm{Cu}_1(J)$. Thus we deduce that $\nu_+(\alpha_{|\mathrm{Cu}_1(I)}) = \alpha_{0|\mathrm{Cu}_1(I)}$. Again, we write ∞_J the maximal element of $\mathrm{Cu}(J)$. Now observe that $\nu_{max}(\alpha_{|I})(x,k) = (\alpha_0(x), \alpha_I(k)) + (\infty_J, 0) = (\infty_J, \alpha_I(k))$. Thus (i) follows.
(ii) Apply ν_+ and ν_{max} to the square of Lemma 3.2.25, combined with the natural isomorphisms of Theorem 3.2.30 and condition (i) above to get the result. □

3.3 Quotients in Cu˜ and exactness of the functor Cu$_1$

Definition 3.3.1. Let S be countably-based positively directed and positively convex Cu˜-semigroup. Let I be an ideal of S. We define the following preorder on S: $x \leq_I y$ if there exists $z \in I$ such that $x \leq z + y$. By antisymmetrizing this preorder, we get an equivalence relation on S, denoted $\sim_I$. We denote by $\overline{x} := [x]_{\sim_I}$.

Lemma 3.3.2. *Let S be countably-based positively directed and positively convex* Cu˜*-semigroup. Let I be an ideal of S. We canonically define $\overline{x} + \overline{y} := \overline{x+y}$ and $\overline{x} \leq \overline{y}$ if $x \leq_I y$. Now define $S/I := (S/\sim_I, +, \leq)$. Then S/I is a countably-based positively directed and positively convex* Cu˜*-semigroup. Also, $S \longrightarrow S/I$ is a surjective* Cu˜*-morphism.*

Proof. Let x, y be in S. It is not hard to check that the sum and ordered considered are well-defined, that is, they do not depend on the representative chosen. Let us show that S/I

equipped with this sum and order is a $\mathrm{PoM}^{\sim}$. Let x_1, x_2 and y_1, y_2 be elements in S such that $\overline{x_1} \leq \overline{x_2}$ and $\overline{y_1} \leq \overline{y_2}$. There exist z_1, z_2 in I such that $x_1 + y_1 \leq x_2 + z_1 + y_2 + z_2$, that is, $\overline{x_1 + y_1} \leq \overline{x_2 + y_2}$. We have shown that $(S/I, +, \leq)$ is a $\mathrm{PoM}^{\sim}$. Also notice that the quotient map $S \longrightarrow S/I$ is naturally a surjective $\mathrm{PoM}^{\sim}$-morphism.

In order to show that $(S/I, +, \leq)$ satisfies the Cuntz axioms, and that $S \longrightarrow S/I$ is a $\mathrm{Cu}^{\sim}$-morphism, we proceed in a similar way as in [4, Section 5.1] for quotients in the category Cu. As the proof works exactly the same way here, we will not get into too many details. This is based on the following two facts:

(1) For any $\overline{x} \leq \overline{y}$ in S/I there exist representatives x, y in S such that $x \leq y$. Indeed we know that there are representatives x, y_1 in S and some $z \in I$ such that $x \leq y_1 + z$. Since $y := (y_1 + z) \sim_I y_1$, the claim is proved.

(2) For any increasing sequence $(\overline{x_k})_k$ in S/I, we can find an increasing sequence of representatives $(x_k)_k$ in S. This uses (1) and the fact that I satisfies (O1). Then $z := \sup\limits_{n\in\mathbb{N}}(\sum\limits_{k=0}^{n} z_k)$, where z_k are the elements obtained from (1), is an element of I. We refer the reader to [4, §5.1.2] for more details.

Let $\overline{x} \in S/I$ and let x be a representative of $\overline{x}$ in S. We know there exists p_x in S such that $x + p_x \geq 0$. Since $0 \in I$, we get that $\overline{x} + \overline{p_x} \geq \overline{0}$, that is, S/I is positively directed.

Lastly, let $\overline{x}, \overline{y} \in S/I$ such that $\overline{x} \leq \overline{y}$ and $0 \leq \overline{y}$. Let x be a representative of $\overline{x}$ and y a representative of $\overline{y}$ in S. Then there are elements $z, w \in I$ such that $x \leq y + z$ and $0 \leq y + w$. Since I is positively directed, there exists $z' \in I$ such that $z + z' \geq 0$. Now observe that $x + w + z' \leq y + z + w + z' = (y + w) + (z + z')$ with $y + w + z + z' \geq 0$. By assumption S is positively convex, hence we have $x + w + z' + y + w + z + z' \geq 0$ and thus and in S/I we obtain $\overline{x} + \overline{y} \geq 0$, as desired. □

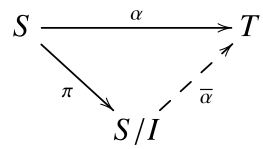
Remark 3.3.3. A priori $(S/I, +, \leq)$ is not positively ordered either. Indeed, one could take for example an algebra that has a non-trivial ideal I with no K_1-obstructions and such that $\mathrm{K}_1(A)$ is not trivial. Then $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I)$ would not be positively ordered.

Lemma 3.3.4. *Let S, T be countably-based positively directed and positively convex* $\mathrm{Cu}^{\sim}$*-semigroups. Let I be an ideal of S. Let $\alpha : S \longrightarrow T$ be a* $\mathrm{Cu}^{\sim}$*-morphism. Suppose that $\alpha(z) = 0$ for any $z \in I$. Then:*

(i) $\alpha(x_1) = \alpha(x_2)$ for any $x_1, x_2 \in S$ such that $\overline{x_1} = \overline{x_2}$ in S/I. We say that α is constant on the classes of S/I.

(ii) There exists a unique $\mathrm{Cu}^{\sim}$*-morphism $\overline{\alpha} : S/I \longrightarrow T$ such that the following diagram is*

commutative:



defined by $\overline{\alpha}(\overline{x}) := \alpha(x)$*, where* $x \in S$ *is any representative of* $\overline{x}$*.*

Proof. By assumption $\alpha(I) = \{0\}$.
(i) Let $x_1, x_2 \in S$ such that $x_1 \sim_I x_2$. Then we know that there exists $z_1, z_2 \in I$ such that $x_1 \leq z_1 + x_2$ and $x_2 \leq z_2 + x_1$. Since $\alpha(z_1) = \alpha(z_2) = 0$, we obtain that $\alpha(x_1) = \alpha(x_2)$.
(ii) Hence we can define $\overline{\alpha} : S/I \longrightarrow T$ by $\overline{\alpha}(\overline{x}) := \alpha(x)$, for any $x \in S$. By construction, the diagram is commutative. We only have to check that $\overline{\alpha}$ is a Cu~-morphism. Using facts (1) and (2) of the proof Lemma 3.3.2, one can check that for any $\overline{x}, \overline{y} \in S/I$ such that $\overline{x} \leq \overline{y}$ (resp $\ll$), there exists representatives x, y in S such that $x \leq y$ (resp $\ll$). Thus we easily obtain that $\overline{\alpha}$ is a Cu~-morphism, which ends the proof. □

3.3.5. In the next theorem, we use the picture of the Cu_1-semigroup described in Proposition 3.1.16.

Theorem 3.3.6. *Let* $A \in C^*$ *and let* $I \in \mathrm{Lat}(A)$*. Let* $\pi : A \longrightarrow A/I$ *be the quotient map. Write* $\pi^* := \mathrm{Cu}_1(\pi) : \mathrm{Cu}_1(A) \longrightarrow \mathrm{Cu}_1(A/I)$*. Then* $\pi^*((x,k)) \leq \pi^*((y,l))$ *if and only if* $(x,k) \leq_{\mathrm{Cu}_1(I)} (y,l)$*. Moreover* π^* *is a surjective* Cu~*-morphism. Thus, it induces a* Cu~*-isomorphism* $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I) \simeq \mathrm{Cu}_1(A/I)$*.*

Proof. Let us start with the surjectivity of π^*. Let $[(a_I, u_I)] \in \mathrm{Cu}_1(A/I)$ where $a_I \in ((A/I)\otimes\mathcal{K})_+$ and u_I is a unitary element of $(\mathrm{her}\, a_I)^\sim$. As π is surjective, we know there exists $a \in A\otimes\mathcal{K}_+$ such that $\pi(a) = a_I$. Moreover, $\mathrm{her}\, a$ has stable rank one, hence unitaries of $(\mathrm{her}\, a_I)^\sim = \pi^\sim(\mathrm{her}\, a^\sim)$ lift. Thus, we can find a unitary element u in $\mathrm{her}\, a^\sim$ such that $\pi^\sim(u) = u_I$. One can then check that $\pi^*([(a,u)]) = [(a_I, u_I)]$.
Let us show the first equivalence of the theorem. Noticing that $\pi^*(\mathrm{Cu}_1(I)) = \{0_{\mathrm{Cu}_1(A/I)}\}$ and that π^* is order-preserving, one easily gets the backward implication.
Now let (x,k) and (y,l) be elements of $\mathrm{Cu}_1(A)$ such that $\pi^*((x,k)) \leq \pi^*((y,l))$. We write $(\overline{x}, \overline{k}) := \pi^*((x,k)) = (\pi_0^*(x), \pi_x^*(k))$ and $(\overline{y}, \overline{l}) := \pi^*((y,l)) = (\pi_0^*(y), \pi_y^*(l))$. Thus we have $\overline{x} \leq \overline{y}$ in $\mathrm{Cu}(A/I)$. By Paragraph 1.3.11, we know that $\mathrm{Cu}(A/I) \simeq \mathrm{Cu}(A)/\mathrm{Cu}(I)$, where the isomorphism is induced by the natural quotient map $\pi : A \longrightarrow A/I$. Therefore, there exists $z \in \mathrm{Cu}(I)$, such that $x \leq y + z$ in $\mathrm{Cu}(A)$. A fortiori, we choose $z := \infty_I$ and we write $y' := y + z$.

Now by Corollary 3.2.31, we obtain the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{K}_1(I_x) & \xrightarrow{\pi^*_{I_x}} & \mathrm{K}_1(I_{\overline{x}}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \delta_{I_xI_{y'}}} & & \Big\downarrow{\scriptstyle \delta_{I_{\overline{x}}I_{\overline{y}}}} & & \\
\mathrm{K}_1(I_z) & \xrightarrow[\delta_{I_zI_{y'}}]{} & \mathrm{K}_1(I_{y'}) & \xrightarrow[\pi^*_{I_{y'}}]{} & \mathrm{K}_1(I_{\overline{y}}) & \longrightarrow & 0
\end{array}$$

Thus, we get on the one hand that $\mathrm{K}_1(I_{y'})/\delta_{I_zI_{y'}}(\mathrm{K}_1(I_z)) \simeq \mathrm{K}_1(I_{\overline{y}})$ and on the other hand $\pi^*_{I_{y'}} \circ \delta_{I_xI_{y'}} = \delta_{I_{\overline{x}}I_{\overline{y}}} \circ \pi^*_{I_x}$. Moreover, by hypothesis, we have $\delta_{I_{\overline{x}}I_{\overline{y}}}(\overline{k}) = \overline{l}$. So one finally gets that $\delta_{I_xI_{y'}}(k) = \delta_{I_yI_{y'}}(l) + \delta_{I_zI_{y'}}(l')$ for some $l' \in \mathrm{K}_1(I_z)$. That is, there exists $(z, l') \in \mathrm{Cu}_1(I)$ such that $(x, k) \leq (y, l) + (z, l')$. This ends the proof of the equivalence.
Finally, we already know that $\mathrm{Cu}_1(I)$ is an ideal of $\mathrm{Cu}_1(A)$ and that $\pi^* : \mathrm{Cu}_1(A) \twoheadrightarrow \mathrm{Cu}_1(A/I)$ is constant on classes of $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I)$. By Lemma 3.3.4, π^* induces a surjective Cu$^\sim$-morphism $\overline{\pi^*} : \mathrm{Cu}_1(A)/\mathrm{Cu}_1(I) \longrightarrow \mathrm{Cu}_1(A/I)$. Furthermore, the equivalence that we have just proved states that $\overline{\pi^*}$ is also an order-embedding. Thus we get a Cu$^\sim$-isomorphism $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I) \simeq \mathrm{Cu}_1(A/I)$. □

Definition 3.3.7. Let S, T and V be countably-based positively directed and positively convex Cu$^\sim$-semigroups. Let $f : S \longrightarrow T$ be a Cu$^\sim$-morphism. We define $\operatorname{im} f := \{(t_1, t_2) \in T \times T : \exists s \in S, t_1 \leq f(s) + t_2\}$ and $\ker f := \{(s_1, s_2) \in S \times S : f(s_1) \leq f(s_2)\}$.
Now consider $g : T \longrightarrow V$ a Cu$^\sim$-morphism. We say that a sequence $... \longrightarrow S \xrightarrow{f} T \xrightarrow{g} V \longrightarrow ...$ is *exact at* T if: $\ker g = \operatorname{im} f$. We say that it is *short-exact* if $0 \longrightarrow S \xrightarrow{f} T \xrightarrow{g} V \longrightarrow 0$ is exact everywhere. Finally, we say that a short-exact sequence is *split*, if there exists a Cu$^\sim$-morphism $q : V \longrightarrow S$ such that $g \circ q = id_V$.

Proposition 3.3.8. *Let* $S \xrightarrow{f} T \xrightarrow{g} V$ *be a sequence in* Cu$^\sim$*, where* S, T, V *and* f, g *are as in Definition 3.3.7. Then:*
(i) f *is an order embedding if and only if* $0 \longrightarrow S \xrightarrow{f} T$ *is exact.*
(ii) If g *is surjective then* $T \xrightarrow{g} V \longrightarrow 0$ *is exact. If moreover* $g(T)$ *is an ideal of* V*, then the converse is true.*

Proof. We recall that for $0 \xrightarrow{0} S$, $\operatorname{im} 0 = \{(s_1, s_2) \in S^2 \mid s_1 \leq s_2\}$ and that for $T \xrightarrow{0} 0$, $\ker 0 = T^2$. Let us consider a sequence $S \xrightarrow{f} T \xrightarrow{g} V$ in Cu$^\sim$.
(i) f is an order-embedding if and only if $[s_1 \leq s_2 \Leftrightarrow f(s_1) \leq f(s_2)]$, that is, if and only if $\operatorname{im} 0 = \ker f$.
(ii) Suppose g is surjective and let v_1, v_2 be elements in V. Since V is countably-based and

positively directed, by Lemma 3.2.10 we know that there exists a unique element v in V_{max} such that $v_2 + v = e_{V_{max}}$. Thus, we have $v_1 \leq e_{V_{max}} + v_1 = v_2 + v + v_1$. By surjectivity, there exists $t \in T$ such that $g(t) = v + v_1$. Hence, for any v_1, v_2 in V there exists $t \in T$ such that $v_1 \leq g(t) + v_2$, that is, $\ker 0 = T^2 = \operatorname{im} g$.
Suppose now that $T \xrightarrow{g} V \longrightarrow 0$ is exact and that $g(T)$ is an ideal of V. We know that for any v_1, v_2, there exists $t \in T$ such that $v_1 \leq g(t) + v_2$. In particular for $v_2 = 0$, we get that for any $v \in V$, there exists $t \in T$ such that $v \leq g(t)$. Moreover $g(T)$ is order-hereditary, hence $v \in g(T)$ and thus g is surjective as desired. □

Lemma 3.3.9. *Let $S \xrightarrow{f} T \xrightarrow{g} V$ be a sequence in* Cu˜*. Also suppose $f(S)$ is an ideal of T and that g is constant on the classes of $T/f(S)$. By Lemma 3.3.4, we can consider $\overline{g} : T/f(S) \longrightarrow V$. If $\overline{g}$ is a* Cu˜*-isomorphism, then $S \xrightarrow{f} T \xrightarrow{g} V \longrightarrow 0$ is exact. If moreover $g(T)$ is an ideal of V, then the converse is true.*

Proof. Suppose $T/f(S) \overset{\overline{g}}{\simeq} V$. Since $\overline{g}$ is an isomorphism, we know that g is surjective. Thus, by Proposition 3.3.8, we get exactness at V. Let us show exactness at T. We have the following equivalences:
$(t_1, t_2) \in \ker g$ if and only if $g(t_1) \leq g(t_2)$ -by definition- if and only if $g(\overline{t_1}) \leq g(\overline{t_2})$ -since g is constant on classes of $T/f(S)$- if and only if $\overline{t_1} \leq \overline{t_2}$ -since $\overline{g}$ is an order-embedding- if and only if $t_1 \leq f(s) + t_2$ for some $s \in S$ -by definition-, that is, if and only if $(t_1, t_2) \in \operatorname{im} f$. □

Theorem 3.3.10. *Let $A \in C^*$ and let $I \in \mathrm{Lat}(A)$. Consider the canonical short exact sequence: $0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0$. Then, the following sequence is short exact in* Cu˜*:*

$$0 \longrightarrow \mathrm{Cu}_1(I) \xrightarrow{i^*} \mathrm{Cu}_1(A) \xrightarrow{\pi^*} \mathrm{Cu}_1(A/I) \longrightarrow 0$$

Proof. By Corollary 3.2.23, we know that $\mathrm{Cu}_1(I)$ is an ideal of $\mathrm{Cu}_1(A)$ and that i^* is an order-embedding. Hence by Proposition 3.3.8 (i), the sequence is exact at $\mathrm{Cu}_1(I)$. From Theorem 3.3.6, we also know that π^* is constant on classes of $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I)$ and that $\overline{\pi^*}$: $\mathrm{Cu}_1(A)/\mathrm{Cu}_1(I) \simeq \mathrm{Cu}_1(A/I)$ is an isomorphism. Thus using Lemma 3.3.9 the result follows. □

Corollary 3.3.11. *For any $A \in C^*$, consider the canonical exact sequence $0 \longrightarrow A \xrightarrow{i} A^\sim \xrightarrow{\pi} A^\sim/A \simeq \mathbb{C} \longrightarrow 0$. Then there is a short exact sequence:*

$$0 \longrightarrow \mathrm{Cu}_1(A) \xrightarrow{i^*} \mathrm{Cu}_1(A^\sim) \xrightarrow{\pi^*} \overline{\mathbb{N}} \times \{0\} \longrightarrow 0$$

where π^* is induced by π.

3.3.12. Now that we have a number of tools regarding ideals and exact sequences in $\mathrm{Cu}^\sim$, we will relate ideals, maximal elements, and positive cones through exact sequences. Recall that for $S \in \mathrm{Cu}^\sim$ countably-based positively directed, we have $S_+ \in \mathrm{Cu}$ and that $S_{max} \in \mathrm{AbGp}$; see Proposition 3.2.8.
Also, a Cu-semigroup (respectively Cu-morphism) can be trivially seen as a $\mathrm{Cu}^\sim$-semigroup since $\mathrm{Cu} \subseteq \mathrm{Cu}^\sim$. The same can be done for any abelian group (respectively any AbGp-morphism), -a fortiori, for the abelian group S_{max} and the AbGp-morphism α_{max}-: Given $G \in \mathrm{AbGp}$, define $g_1 \leq g_2$ if and only if $g_1 = g_2$. From this, it follows that also $g_1 \ll g_2$ if and only if $g_1 = g_2$. This defines a functor $\mathrm{AbGp} \longrightarrow \mathrm{Cu}^\sim$ which allows us to see the category AbGp as a subcategory of Cu.
Therefore, in what follows, we consider ν_+ and ν_{max} as functors with codomain $\mathrm{Cu}^\sim$. Finally, note that all of the proofs will be done in an abstract setting. Further, by Theorem 3.2.30, we will be able to directly apply those results to $\mathrm{Cu}(A)$ and $\mathrm{K}_1(A)$, also seen as $\mathrm{Cu}^\sim$-semigroups.

Definition 3.3.13. Let S be a countably-based and positively directed $\mathrm{Cu}^\sim$-semigroup. Let us define two $\mathrm{Cu}^\sim$-morphisms that link S to S_+ on the one hand and to S_{max} on the other hand, as follows:

$$\begin{aligned} i : S_+ &\overset{\subseteq}{\hookrightarrow} S \\ s &\longmapsto s \end{aligned} \qquad\qquad \begin{aligned} j : S &\twoheadrightarrow S_{max} \\ s &\longmapsto s + e_{S_{max}} \end{aligned}$$

Proposition 3.3.14. *Let S be a countably-based and positively directed* $\mathrm{Cu}^\sim$*-semigroup. Consider the* $\mathrm{Cu}^\sim$*-morphisms defined in Definition 3.3.13, then i is an order-embedding, and j is surjective. Moreover, the following sequence in* $\mathrm{Cu}^\sim$ *is split-exact:*

$$0 \longrightarrow S_+ \xrightarrow{\ i\ } S \underset{\underset{q}{\longleftarrow}}{\xrightarrow{\ j\ }} S_{max} \longrightarrow 0$$

where the split morphism is defined by $q(s) := s$.

Proof. It is trivial to check that i is a well-defined order-embedding $\mathrm{Cu}^\sim$-morphism. We now need to check whether j is a well-defined additive map. From Lemma 3.2.10, we know that $s + e_{S_{max}} \in S_{max}$, for any $s \in S$. Also, because $2.e_{S_{max}} = e_{S_{max}}$, we get that j is additive. Further, whenever $s \leq s'$, we know that $s + e_{S_{max}} \leq s' + e_{S_{max}}$. Since $s + e_{S_{max}} \in S_{max}$, we deduce that $j(s) = j(s')$ whenever $s \leq s'$. Further, $j(0) = e_{S_{max}}$. Thus, j is a surjective $\mathrm{Cu}^\sim$-morphism.
By Proposition 3.3.8, we get exactness of the sequence at S_+ and S_{max}. Now let us check that the sequence is exact at S. Let $(s_1, s_2) \in \ker j$. Hence $j(s_1) = j(s_2)$, that is, $s_1 + e_{S_{max}} = s_2 + e_{S_{max}}$. Since $e_{S_{max}} \in S_+$, we easily get that $s_1 \leq s_1 + e_{S_{max}} = s_2 + e_{S_{max}}$, which proves that

ker $j \subseteq$ im i. Conversely, let $(s_1, s_2) \in$ im j. Then we know that there exists a positive element $s \in S_+$ such that $s_1 \leq s + s_2$. Since $e_{S_{max}}$ is the maximal positive element of S, we can take $s = e_{S_{max}}$. Then we easily get that $j(s_1) \leq j(s_2)$ -in fact, they are equal-. Thus we conclude that im i = ker j, which ends the proof. □

Remark 3.3.15. Note that we could not have used Lemma 3.3.9 here, since S_+ is not a Cu~ ideal of S. Indeed the smallest ideal containing S_+ is S itself.

Theorem 3.3.16. *Let S, T be countably-based and positively directed* Cu~*-semigroups. Let $\alpha : S \longrightarrow T$ be a* Cu~ *morphism. Viewing the functors ν_+ and ν_{max} with codomain* Cu~*, as in Paragraph 3.3.12, we obtain the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_+ & \xrightarrow{i} & S & \xrightarrow{j} & S_{max} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_+} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_{max}} & & \\ 0 & \longrightarrow & T_+ & \xrightarrow[i]{} & T & \xrightarrow[j]{} & T_{max} & \longrightarrow & 0 \end{array}$$

Furthermore, if α is a Cu~*-isomorphism, then α_+ is a* Cu*-isomorphism and α_{max} is an abelian group isomorphism.*

Proof. We know from Proposition 3.3.14 that the row sequences are split-exact. Besides $\alpha_+ = \alpha_{|S_+}$ hence the left-square is commutative. Now take any $s \in S$. we have $\alpha_{max} \circ j_S(s) = \alpha_{max}(s + e_{S_{max}}) = \alpha(s) + 2e_{T_{max}} = \alpha(s) + e_{T_{max}} = j_T \circ \alpha(s)$, which proves that the right-square is commutative.
Now assume that α is an isomorphism. By functoriality of ν_+ and ν_{max}, we obtain that α_+ is a Cu-isomorphism whose inverse is $(\alpha^{-1})_+$ and that α_{max} is an abelian group isomorphism whose inverse is $(\alpha^{-1})_{max}$. □

Corollary 3.3.17. *Let S, T and α be as in Theorem 3.3.16. Assume also that S, T are positively convex. Let I be an ideal of S and $J := I_{\alpha(e_{I_{max}})}$, the smallest ideal of T containing $\alpha(I)$ (see Lemma 3.2.25). We obtain the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_+ & \xrightarrow{i} & I & \xrightarrow{j} & I_{max} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\alpha_{|I})_+} & & \downarrow{\scriptstyle \alpha_{|I}} & & \downarrow{\scriptstyle (\alpha_{|I})_{max}} & & \\ 0 & \longrightarrow & J_+ & \xrightarrow[i]{} & J & \xrightarrow[j]{} & J_{max} & \longrightarrow & 0 \end{array}$$

Furthermore, if α is a Cu~*-isomorphism, then $\alpha(I) = J$ and $\alpha_{|I} : I \longrightarrow J$ is a* Cu~*-isomorphism.*

A fortiori, we also have $(\alpha_{|I})_+ : I_+ \longrightarrow J_+$ *is a* Cu*-isomorphism and* $\alpha_I : I_{max} \longrightarrow J_{max}$ *is an abelian group isomorphism.*

Proof. We only have to check that whenever α is an isomorphism, then $J = \alpha(I)$ and that $\alpha_{|I} : I \longrightarrow J$ defined as in Lemma 3.2.25 is an isomorphism. Then the conclusion will follow applying Theorem 3.3.16 to $\alpha_{|I}$. Suppose that α is a $\mathrm{Cu}^{\sim}$-isomorphism. We know that $\alpha_{|I} : I \longrightarrow J$ sends any element $x \in I$ to $\alpha(x) \in J$. Since α is an order-embedding, so is $\alpha_{|I}$. By Lemma 3.2.25, we know that $\alpha(I) \subseteq J$ and that $\alpha(e_{I_{max}}) = e_{J_{max}}$. Now since α is an isomorphism, we obtain that $\alpha^{-1}(e_{J_{max}}) = e_{I_{max}}$. That is, by Lemma 3.2.25, $\alpha^{-1}(J) \subseteq I$. We deduce that $\alpha(I) = J$ and that α_I is a $\mathrm{Cu}^{\sim}$-isomorphism. □

3.3.18. As observed in Paragraph 3.3.12, we can use Theorem 3.2.30, to obtain the following results in the category C^*:

Theorem 3.3.19. *Let* $A, B \in C^*$. *Let* $\phi : A \longrightarrow B$ *be a* **-homomorphism. Then the following diagram is commutative with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Cu}(A) & \xrightarrow{i} & \mathrm{Cu}_1(A) & \xrightarrow{j} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Cu}(\phi)} & & \downarrow{\scriptstyle \mathrm{Cu}_1(\phi)} & & \downarrow{\scriptstyle \mathrm{K}_1(\phi)} & & \\
0 & \longrightarrow & \mathrm{Cu}(B) & \xrightarrow{i} & \mathrm{Cu}_1(B) & \xrightarrow{j} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

Furthermore, if $\mathrm{Cu}_1(\phi)$ *is a* $\mathrm{Cu}^{\sim}$*-isomorphism, then* $\mathrm{Cu}(\phi)$ *is a* Cu*-isomorphism and* $\mathrm{K}_1(\phi)$ *is a* AbGp*-isomorphism.*

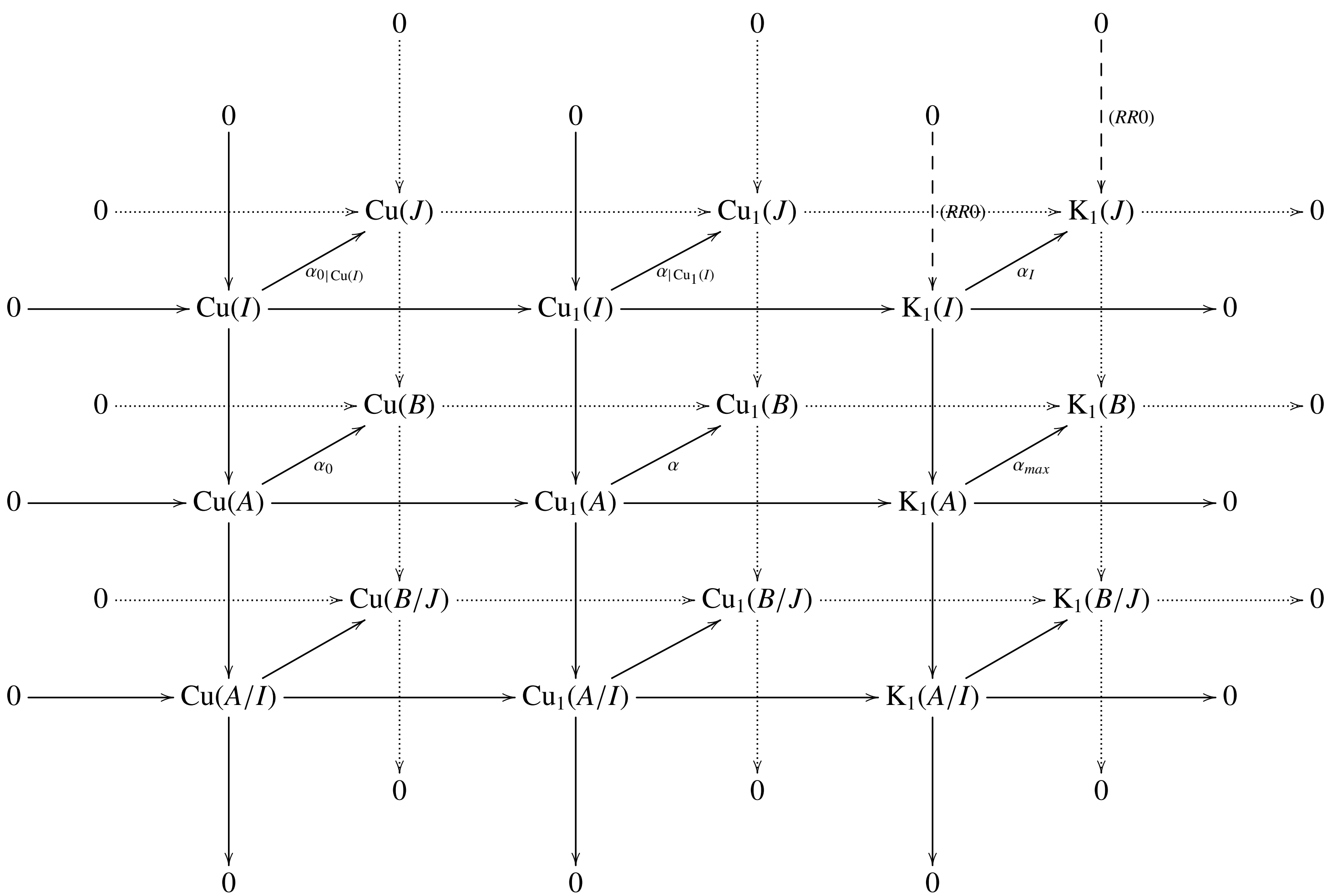
Let $I \in \mathrm{Lat}(A)$. *Write* $J := \overline{B\phi(I)B}$, *the smallest ideal of* B *containing* $\phi(I)$ *and* $\alpha := \mathrm{Cu}_1(\phi)$. *Let us use the same notations as in Lemma 3.1.19, that is,* $\alpha = (\alpha_0, \{\alpha_I\}_{I\in\mathrm{Lat}(A)})$. *Then the following diagram is commutative with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Cu}(I) & \xrightarrow{i} & \mathrm{Cu}_1(I) & \xrightarrow{j} & \mathrm{K}_1(I) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{0|\mathrm{Cu}(I)}} & & \downarrow{\scriptstyle \alpha_{|\mathrm{Cu}_1(I)}} & & \downarrow{\scriptstyle \alpha_I} & & \\
0 & \longrightarrow & \mathrm{Cu}(J) & \xrightarrow[i]{} & \mathrm{Cu}_1(J) & \xrightarrow[j]{} & \mathrm{K}_1(J) & \longrightarrow & 0
\end{array}$$

Furthermore, if α *is a* $\mathrm{Cu}^{\sim}$*-isomorphism, then* $\alpha(\mathrm{Cu}_1(I)) = \mathrm{Cu}_1(J)$ *and* $\alpha_{|\mathrm{Cu}_1(I)} : \mathrm{Cu}_1(I) \longrightarrow \mathrm{Cu}_1(J)$ *is a* $\mathrm{Cu}^{\sim}$*-isomorphism. A fortiori, we also have* $\alpha_{0|\mathrm{Cu}(I)} : \mathrm{Cu}(I) \longrightarrow \mathrm{Cu}(J)$ *is a* Cu*-isomorphism and* $\alpha_I : \mathrm{K}_1(I) \longrightarrow \mathrm{K}_1(J)$ *is a* AbGp*-isomorphism.*

Proof. Combine Theorem 3.3.16 and Corollary 3.3.17 with Lemma 3.2.25. □

3.3.20. Let us summarize our exactness results. For any $A, B \in C^*$, any $I \in \mathrm{Lat}(A)$ and any $J \in \mathrm{Lat}(B)$, we have the following commutative exact diagram:



where the vertical exact sequences are those obtained passing the canonical short-exact sequence of an ideal through their respective functors, and the horizontal sequences are those described in Theorem 3.3.19.

Chapter 4

Computation of Cu_1-semigroups

In this chapter, the aim is to compute the Cu_1-semigroup in some specific settings. In the process, we will remind the reader about well-know constructions, such as UHF C^*-algebras, or NCCW 1, among others. Furthermore, since the Cu_1-semigroup aims to capture more information than $\mathrm{K}_0, \mathrm{K}_1$ and Cu, we will also recall the computation of those in some examples.

4.1 The simple case

Proposition 4.1.1. *Let A be a simple C^*-algebra. Then* $\mathrm{Cu}_1(A)$ *can be described in terms of* $\mathrm{Cu}(A)$ *and* $\mathrm{K}_1(A)$ *as follows:*

$$\begin{aligned} \mathrm{Cu}_1(A) &\overset{\simeq}{\longrightarrow} (\mathrm{Cu}(A)_* \times \mathrm{K}_1(A)) \sqcup \{0\} \\ (x,k) &\longmapsto \begin{cases} 0 \text{ if } x = 0 \\ (x,k) \text{ otherwise} \end{cases} \end{aligned}$$

Proof. Since A is simple, we know that $\mathrm{Lat}(A) = \{0, A\}$. Therefore, in the description of the Cu_1-semigroup of Proposition 3.1.16, we have $\mathrm{Cu}_f(\{0\}) = \{0\}$ and $\mathrm{Cu}_f(A) = \mathrm{Cu}(A)_*$. The result follows. □

4.2 The case of no K_1-obstructions

Definition 4.2.1. We say that a C^*-algebra A has *no* K_1*-obtructions*, if A has stable rank one and $\mathrm{K}_1(I)$ is trivial for any $I \in \mathrm{Lat}(A)$.

4.2.2. An *approximate finite dimensional algebra*, written AF algebra, is an inductive limit of finite direct sums of full matrix algebras. They are completely classified by their K_0 group (see [24]). Actually they were the first class of C^*-algebras classified in the Elliott's classification program. Let us recall some well-known facts:
(i) Any AF algebra A has no K_1 obstructions, that is, $\mathrm{K}_1(I)$ is trivial for any $I \in \mathrm{Lat}(A)$. In particular $\mathrm{K}_1(A) \simeq \{0\}$.
(ii) Any AF algebra A has stable rank one.
(iii) They are completely classified by the 3-tuple $(\mathrm{K}_0, \mathrm{K}_{0+}, [1])$ (in the unital case).

Proposition 4.2.3. *Let A be a C^*-algebra with no K_1-obstructions. Then $\mathrm{Cu}_1(A) \simeq \mathrm{Cu}(A)$. In particular, for any* AF *algebra A, $\mathrm{Cu}_1(A) \simeq \mathrm{Cu}(A)$.*

Proof. By assumption, we know that $\mathrm{K}_1(I)$ is trivial for any $I \in \mathrm{Lat}(A)$. Therefore, using again the description of the Cu_1-semigroup of Proposition 3.1.16, we have $\mathrm{Cu}_1(A) \simeq \mathrm{Cu}(A) \times \{0\}$. The result follows. □

4.2.4. Let us end this section by reminding the reader about a specific subclass of AF algebras, that will be used later on: The UHF algebras; see e.g [33],[71, Example 4.6].

Definition 4.2.5. Let $(p_k)_{k\in\mathbb{N}}$ be an enumeration of the prime numbers ($p_0 = 2$). Any natural number n can be uniquely written as a finite product $n := \prod\limits_{k=0}^{l} p_k^{m_k}, m_k \in \mathbb{N}$ of powers of prime numbers. We say that m_k is the multiplicity of the prime p_k in n.
We define a *supernatural number q* as a formal product $q := \prod\limits_{k\in\mathbb{N}} p_k^{m_k}, m_k \in \overline{\mathbb{N}}$, of powers of prime numbers where the multiplicities can be infinite. Note that natural numbers can be identified with supernatural numbers such that $\sum\limits_{k\in\mathbb{N}} m_k < \infty$.

Theorem 4.2.6. *[33, Theorem 1.12] Let q be a supernatural number and let $(q_n)_{n\in\mathbb{N}}$ be a sequence of prime numbers such that $q = \prod\limits_{n\in\mathbb{N}} q_n$. For any $n \in \mathbb{N}$, define $A_n := \bigotimes\limits_{k=0}^{n} M_{q_k}$ and $\phi_{n(n+1)} : A_n \longrightarrow A_n \otimes M_{q_{n+1}}$ that sends $a \longmapsto a \otimes 1_{q_{n+1}}$. The inductive limit of $(A_n, \phi_{nm})_n$ is a well-defined unital simple* AF *algebra. Moreover, since it does not depend on the sequence $(q_n)_n$ chosen, we set $M_q := \varinjlim(A_n, \phi_{nm})$. We say that M_q is a uniformly hyperfinite algebra, also written UHF algebra. These algebras are completely classified by their supernatural numbers.*

Proposition 4.2.7. *(see e.g [71, Example 4.6])*
(i) $M_q \otimes M_r \simeq M_{qr}$

(ii) Whenever $q^2 = q$ *then* $m_k \in \{0, \infty\}$. *We say that* M_q *is of infinite type and in this case* $M_q \otimes M_q \simeq M_q$
(iii) $\mathrm{K}_0(M_q) \simeq \mathbb{Z}[\frac{1}{q}] := \{k/l, k \in \mathbb{Z}, l|q\}$ *and* $\mathrm{K}_1(M_q) \simeq 0$
(iv) $\mathrm{Cu}(M_q) \simeq \gamma(\mathbb{N}[\frac{1}{q}] \sqcup \{\infty\})$, *where* γ *is the* Cu*-completion from* PoM *to* Cu. *In particular, for any p prime natural number,* $\mathrm{Cu}(M_{p^\infty}) \simeq \mathbb{N}[\frac{1}{p}] \sqcup]0, \infty]$. *See [3, Chatper 4-(21)].*
(v) Any UHF algebra has a unique trace.

Remark 4.2.8. For $q := p^\infty$, with p a prime natural number, then $\mathbb{Z}[\frac{1}{q}] = \mathbb{Z}[\frac{1}{p}]$ are the well-known *p-adic numbers*.

4.3 AI and A$\mathbb{T}$ algebras: The case of $C([0,1])$ and $C(\mathbb{T})$

4.3.1. An *approximate interval* algebra, written AI algebra, is an inductive limit of finite direct sums of interval algebras, namely C^*-algebras of the form $C([0,1]) \otimes M_n \simeq C([0,1], M_n)$. Analogously, we say that a C^*-algebra is an *approximate circle* algebra, written A$\mathbb{T}$ algebra, is an inductive limit of direct sums of circle algebras, namely C^*-algebras of the form $C(\mathbb{T}) \otimes M_n \simeq C(\mathbb{T}, M_n)$. These classes have been studied extensively over the years (see [19], [26], [64], [42] for instance). Among those:
(i) AI and A$\mathbb{T}$ algebras have stable rank one.
(ii) AI algebras have trivial K_1 group and are completely classified by means of their Cu-semigroup, see [64], [19] (equivalently by means of tracial data and K_0 group, see [73]).
(iii) A$\mathbb{T}$ algebras have torsion-free K_1 group and under certain hypothesis are classified by means of their K_* group, see Chapter 5 for more details.
(iv) Any AI with real rank 0 is an AF algebra. Any A$\mathbb{T}$ algebra with trivial K_1 is an AI algebra.
The point now is to try and compute the Cu_1-semigroup of some C^*-algebras in this class. Therefore, we will first recall some facts before computing explicitly the Cu_1-semigroup of $C(X)$, for $X = [0,1]$ or $X = \mathbb{T}$.

Definition 4.3.2. Let X be a compact metric space (hence Hausdorff and second countable) of covering dimension 1, see Paragraph 6.1.2. Since $C(X)$ is a commutative C^*-algebra observe that for any $f \in C(X)_+$, we have $I_f = \operatorname{her} f$, where I_f is the ideal generated by f; (see Paragraph 1.1.6). Also, it is well-known that: (see e.g [62, Theorem 1],[3, Theorem 3.4])

$$\begin{aligned} \mathrm{Cu}(C(X)) &\overset{\simeq}{\longrightarrow} \mathrm{Lsc}(X, \overline{\mathbb{N}}) \\ [f] &\longmapsto (t \longmapsto [f(t)]) \end{aligned}$$

Moreover, any $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ can be (uniquely) described by a $\subseteq$-decreasing sequence of open sets in X. Indeed, $f = \sum_{n=0}^{\infty} U_n$, where $U_n := f^{-1}(]n; +\infty])$. We say that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is generated by* $\{1_{|U}\}_{U \subseteq X}$.

Definition 4.3.3. Let $f \in C(X)_+$. We define the *support* of f as $\mathrm{supp} f := \{t \in X, f(t) \neq 0\}$. It is an open subset of X. Analogously, we define the *support* of $[f]$ as $\mathrm{supp}[f] := \{t \in X, [f(t)] \neq 0\}$.

Proposition 4.3.4. *Let* $[f], [g] \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. *We recall that we denote* $I_{[f]}$ *the ideal of* $\mathrm{Cu}(C(X))$ *generated by* $[f]$ *(see Paragraph 1.3.10). Then, we have the following:*
(i) $I_{[f]} = \mathrm{Lsc}(\mathrm{supp}[f], \overline{\mathbb{N}})$.
(ii) If $[f] \leq [g]$ *then* $\mathrm{supp}[f] \subseteq \mathrm{supp}[g]$. *The converse holds whenever f and g are elements of* $C(X)_+$. *In that case,* $I_f = \mathrm{her} f = C_0(\mathrm{supp} f)$.
(iii) $\mathrm{supp}[f] \subseteq \mathrm{supp}[g]$ *if and only if* $I_{[f]} \subseteq I_{[g]}$.

Proof. Let $[f] \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. Since $I_{[f]} := \{[g] \in \mathrm{Lsc}(X, \overline{\mathbb{N}}) \mid [g] \leq \infty.[f]\}$ and $\leq$ is point-wise in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ (i) and (iii) follow. (ii) is proved in [6, Proposition 2.5]. □

Corollary 4.3.5. *The open subsets of X, that we write $\mathcal{O}(X)$, are in one-to-one correspondence with the ideals of* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. *In fact, we have the following bijection:*

$$\begin{aligned} \mathcal{O}(X) &\simeq \mathrm{Lat}(\mathrm{Lsc}(X, \overline{\mathbb{N}})) \\ U &\longmapsto I_{1_{|U}} \\ \mathrm{supp}(\infty_I) &\longleftarrow I \end{aligned}$$

where ∞_I is the largest element of I (see Paragraph 1.3.10, Definition 3.1.10). Thus, any ideal of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *is countably-based.*

4.3.6. Let us restrict ourselves to the interval and the circle. That is, $X = [0,1]$ or $X = \mathbb{T}$. Let $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. Since $\mathrm{supp} f$ is an open subset of X, it can uniquely be described as a disjoint union of at most countably many open arcs of X. That is, $\mathrm{supp} f = \bigcup_{i=1}^{n_f} U_i$, for some $n_f \in \mathbb{N}$, where U_i are pairwise disjoint open arcs of X. In what follows, we recall that $\mathrm{Cu}_f(I)$ are the elements of $\mathrm{Cu}(A)$ full in $\mathrm{Cu}(I)$, for any C^*-algebra A and any $I \in \mathrm{Lat}(A)$, see Definition 3.1.10. Also we choose the following convention: $\bigoplus_1^{-1} \mathbb{Z} = \bigoplus_1^{0} \mathbb{Z} = \{0\}$

4.3.7. The $C([0,1])$ case.

Lemma 4.3.8. *Let* $I \in \mathrm{Lat}(C([0,1]))$. *Consider* $U := \mathrm{supp}(\infty_I)$*, the unique open set of* $[0,1]$ *that corresponds to* I. *We have:*
(i) $\mathrm{Cu}_f(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*)$.
(ii) $\mathrm{Cu}_f(I) \times \mathrm{K}_1(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*) \times (\overset{m_U}{\underset{1}{\oplus}} \mathbb{Z})$*, where* $m_U := n_{(1_{|U})} - (1_U(0) + 1_U(1))$.

Proof. By Proposition 4.3.4, we know that $\mathrm{Cu}(I) = I_{1_{|U}} \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}})$ and using Proposition 4.3.4 (iii) we obtain that $\mathrm{Cu}_f(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*)$. Furthermore, write $\mathrm{supp}\, 1_{|U} = \bigcup_{i=1}^{n_{(1_{|U})}} U_i$ as in Paragraph 4.3.6. Since the open arcs of $[0,1]$ are of the following forms:

$$]a,b[\quad [0,1] \quad]a,1] \quad [0,a[\quad \emptyset,$$

and since the K_1 groups of the C^*-algebras constructed as continuous map over those open arcs are respectively the following:

$$\mathbb{Z} \qquad \{0\} \qquad \{0\} \qquad \{0\} \qquad \{0\},$$

the result follows. □

Theorem 4.3.9. *Let* $V_0 := [0,1[$ *and* $V_1 :=]0,1]$. *Then:*
(i)

$$\begin{aligned}\mathrm{Cu}_1(C([0,1])) &\simeq \bigsqcup_{U \in O([0,1])} \mathrm{Lsc}(U, \overline{\mathbb{N}}_*) \times (\overset{m_U}{\underset{1}{\oplus}} \mathbb{Z}) \\ &\simeq \mathrm{Cu}_1(C(]0,1[)) \sqcup (\bigsqcup_{i=0,1} \mathrm{Lsc}(V_i, \overline{\mathbb{N}}_*) \times \{0\}) \sqcup \mathrm{Lsc}([0,1], \overline{\mathbb{N}}_*) \times \{0\}.\end{aligned}$$

(ii) $\mathrm{Cu}_1(C([0,1]))/\,\mathrm{Cu}_1(C_0(]0,1])) \simeq \overline{\mathbb{N}} \times \{0\}$.
(iii) $\mathrm{Cu}_1(C([0,1]))_c \simeq (\{n.1_{|[0,1]}\}_{n \in \overline{\mathbb{N}}}) \times \{0\}$.

Proof. (i) Combine Proposition 3.1.16 with Lemma 4.3.8 and Corollary 4.3.5.
(ii) Since $C_0(]0,1])^\sim \simeq C([0,1])$, we get the result by Corollary 3.3.11.
(iii) From Corollary 2.1.15, we know that $(x,k) \in \mathrm{Cu}_1(C([0,1]))$ is a compact element if and only if x is compact in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ if and only if x is constant on $[0,1]$. □

4.3.10. The $C(\mathbb{T})$ case.

Lemma 4.3.11. *Let* $I \in \mathrm{Lat}(C(\mathbb{T}))$. *Consider* $U := \mathrm{supp}(\infty_I)$*, the unique open set of* $\mathbb{T}$ *that corresponds to* I. *We have:*
(i) $\mathrm{Cu}_f(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*)$.
(ii) $\mathrm{Cu}_f(I) \times \mathrm{K}_1(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*) \times (\overset{n_U}{\underset{1}{\oplus}} \mathbb{Z})$*, where* $n_U := n_{(1_{|U})}$.

Proof. By Proposition 4.3.4, we know that $\mathrm{Cu}(I) = I_{1_{|U}} \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}})$ and using Proposition 4.3.4 (iii) we obtain that $\mathrm{Cu}_f(I) \simeq \mathrm{Lsc}(U, \overline{\mathbb{N}}_*)$. Furthermore, write $\operatorname{supp} 1_{|U} = \bigcup\limits_{i=1}^{n(1_{|U})} U_i$ as in Paragraph 4.3.6. Since the open arcs of $\mathbb{T}$ are of the following forms:

$$]a,b[\qquad \mathbb{T} \qquad \emptyset,$$

and since the K_1 groups of the C^*-algebras constructed as continuous map over those open arcs are respectively the following:

$$\mathbb{Z} \qquad \mathbb{Z} \qquad \{0\},$$

the result follows. □

Theorem 4.3.12. *We have*

(i)
$$\begin{aligned}\mathrm{Cu}_1(C(\mathbb{T})) &\simeq \bigsqcup_{U \in \mathcal{O}(\mathbb{T}))} \mathrm{Lsc}(U, \overline{\mathbb{N}}_*) \times \bigoplus_1^{n_U} \mathbb{Z} \\ &\simeq \mathrm{Cu}_1(C(]0,1[)) \sqcup \mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}_*) \times \mathbb{Z}.\end{aligned}$$

(ii) $\mathrm{Cu}_1(C(\mathbb{T})) / \mathrm{Cu}_1(C_0(]0,1[)) \simeq \overline{\mathbb{N}} \times \{0\}$.

(iii) $\mathrm{Cu}_1(C(\mathbb{T}))_c \simeq (\{n.1_{|\mathbb{T}}\}_{n \in \overline{\mathbb{N}}}) \times \mathbb{Z}$.

Proof. (i) Combine Proposition 3.1.16 with Lemma 4.3.8 and Corollary 4.3.5.
(ii) Since $C_0(]0,1[)^\sim \simeq C(\mathbb{T})$, we get the result by Corollary 3.3.11.
(iii) From Corollary 2.1.15, we know that $(x,k) \in \mathrm{Cu}_1(C(\mathbb{T}))$ is a compact element if and only if x is compact in $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ if and only if x is constant on $\mathbb{T}$. □

4.3.13. Now that we have computed the Cu_1-semigroup of the interval algebra and the circle algebra, we are able to obtain the Cu_1-semigroup of any AI and A$\mathbb{T}$ algebra, by computing inductive limits in $\mathrm{Cu}^\sim$; see Corollary 2.3.10. Actually, we will next compute a concrete example of an an A$\mathbb{T}$ algebra that is constructed as $C(\mathbb{T})\otimes$UHF (respectively an AI algebra that can be constructed as $C([0,1])\otimes$UHF).

4.3.14. Let q be a supernatural number and consider M_q the UHF algebra associated to q. Consider any sequence of prime numbers $(q_n)_n$ such that $q = \prod\limits_{n \in \mathbb{N}} q_n$. Write $(A_n, \phi_{nm})_n$ the inductive system associated to $(q_n)_n$ as in Theorem 4.2.6. Now consider the following A$\mathbb{T}$ algebra: $A := \varinjlim_n (C(\mathbb{T}) \otimes A_n, id \otimes \phi_{nm})$. In fact, $A \simeq C(\mathbb{T}) \otimes M_q$ (a similar construction can be done for the interval).

Theorem 4.3.15. *Let* M_q *be a UHF algebra and let* $V_0 := [0,1[$ *and* $V_1 :=]0,1]$. *Then:*
(i) $\mathrm{Cu}_1(C(\mathbb{T}) \otimes M_q) \simeq \bigsqcup_{U \in \mathcal{O}(\mathbb{T})} \mathrm{Lsc}(U, \mathrm{Cu}(M_q)_*) \times (\overset{n_U}{\underset{1}{\oplus}} \mathrm{K}_0(M_q))$.
(ii) $\mathrm{Cu}_1(C([0,1]) \otimes M_q) \simeq \bigsqcup_{U \in \mathcal{O}(]0,1[)} \mathrm{Lsc}(U, \mathrm{Cu}(M_q)_*) \times (\overset{m_U}{\underset{1}{\oplus}} \mathrm{K}_0(M_q)) \sqcup (\bigsqcup_{i=0,1} \mathrm{Lsc}(V_i, \mathrm{Cu}(M_q)_*) \times \{0\}) \sqcup \mathrm{Lsc}([0,1], \mathrm{Cu}(M_q)_*) \times \{0\}$.

In particular, for any UHF algebra of infinite type M_{p^∞}, *we get:*
(i) $\mathrm{Cu}_1(C(\mathbb{T}) \otimes M_q) \simeq \bigsqcup_{U \in \mathcal{O}(\mathbb{T})} \mathrm{Lsc}(U, \mathbb{N}[\frac{1}{p}]_* \sqcup]0,\infty]) \times (\overset{n_U}{\underset{1}{\oplus}} \mathbb{Z}[\frac{1}{p}])$.
(ii) $\mathrm{Cu}_1(C([0,1]) \otimes M_q) \simeq \bigsqcup_{U \in \mathcal{O}(]0,1[)} \mathrm{Lsc}(U, \mathbb{N}[\frac{1}{p}]_* \sqcup]0,\infty]) \times (\overset{m_U}{\underset{1}{\oplus}} \mathbb{Z}[\frac{1}{p}]) \sqcup (\bigsqcup_{i=0,1} \mathrm{Lsc}(V_i, \mathbb{N}[\frac{1}{p}]_* \sqcup]0,\infty]) \times \{0\}) \sqcup \mathrm{Lsc}([0,1], \mathbb{N}[\frac{1}{p}]_* \sqcup]0,\infty]) \times \{0\}$.

Proof. We will only compute the circle case as the interval case is done similarly.
Since UHF algebras are simple, we know that all ideals of $C(\mathbb{T}) \otimes M_q$ are of the form $C_0(U) \otimes M_q$ for some $U \in \mathcal{O}(\mathbb{T})$. Hence, using Künneth formula (see Theorem 1.1.18), we obtain that $\mathrm{K}_1(C_0(U) \otimes M_q) \simeq (\overset{n_U}{\underset{1}{\oplus}} \mathbb{Z}) \otimes \mathrm{K}_0(M_q) \simeq \overset{n_U}{\underset{1}{\oplus}} \mathrm{K}_0(M_q)$. On the other hand, by Theorem 1.3.15, we compute that $\mathrm{Cu}(C_0(U) \otimes M_q) \simeq \mathrm{Lsc}(U, \mathrm{Cu}(M_q))$. The result follows from Proposition 3.1.16. □

4.4 The NCCW 1 complexes

4.4.1. In this section, we will be interested in a more general class: the NCCW 1 complexes. We refer the reader to [63] for a classification of some of these C^*-algebras by means of $\mathrm{Cu}^\sim$, an augmented version of the Cuntz semigroup.
First, we will recall the general definition of NCCW 1 complexes, recall the compute of their K-theory and their Cuntz semigroup. Observe that NCCW 1 algebras contain the AF, AI, A$\mathbb{T}$ and AH_d algebras. We will define the latter and specify a bit more about their classification that has been widely studied (see e.g [25],[26],[42]) in the next chapter (see Chapter 5). Finally, we will see that a priori Cu_1 does not pass through pullbacks, and hence does not allow a direct computation from the construction of the C^*-algebra.

Definition 4.4.2. Let E, F be finite dimensional C^*-algebras and let $\phi_0, \phi_1 : E \longrightarrow$ be two *-homomorphisms. We define a *non-commutative CW complex of dimension 1*, written NCCW 1, as the following pullback:

$$\begin{array}{ccc} A & \longrightarrow & C([0,1],F) \\ \downarrow & & \downarrow{\scriptstyle (ev_0, ev_1)} \\ E & \xrightarrow[(\phi_0,\phi_1)]{} & F \oplus F \end{array}$$

We write such a pullback as $A := A(E, F, \phi_0, \phi_1)$ and refer to the class of inductive limits of finite direct sums of NCCW 1 as NCCW 1 *algebras*.

Lemma 4.4.3. *Any* $A \in$ NCCW 1 *has stable rank one.*

Proof. Let $A := A(E, F, \phi_0, \phi_1)$ be a pullback as constructed above. Since the map (ev_0, ev_1) is surjective and since both E and $C([0, 1], F)$ have stable rank one, using [16, Theorem 4.1], we deduce that A has stable rank one. The result follows for any $A \in$ NCCW 1 as stable rank one passes through direct sums and inductive limits (see e.g. [60, Theorem 5.1]). □

Proposition 4.4.4. *Let* $A := A(E, F, \phi_0, \phi_1)$ *be a* NCCW 1 *complex and let* p, l *be the number of terms in the finite direct sum of* E, F *respectively. We write* $\alpha := \mathrm{K}_0(\phi_0) : \mathbb{Z}^p \longrightarrow \mathbb{Z}^l$ *and* $\beta := \mathrm{K}_0(\phi_1) : \mathbb{Z}^p \longrightarrow \mathbb{Z}^l$*. Finally we write* $\gamma_0 := \mathrm{Cu}(\phi_0) : \overline{\mathbb{N}}^p \longrightarrow \overline{\mathbb{N}}^l$ *and* $\delta_0 := \mathrm{Cu}(\phi_1) : \overline{\mathbb{N}}^p \longrightarrow \overline{\mathbb{N}}^l$*. Then:*
(i) $\mathrm{K}_0(A) \simeq \ker(\alpha - \beta) \subseteq \mathbb{Z}^p$.
(ii) $\mathrm{K}_1(A) \simeq \mathbb{Z}^l / \operatorname{im}(\alpha - \beta)$.
(iii) $\mathrm{Cu}(A) \simeq \{(f, b) \in \mathrm{Lsc}([0, 1], \overline{\mathbb{N}}^l) \oplus \overline{\mathbb{N}}^p \mid f(0) = \gamma_0(b), f(1) = \delta_0(b)\}$.

Proof. The proof of (i) and (ii) can be found for instance in [59, §2.4]. We still give a sketch here:
Define $SF := C_0(]0, 1[) \otimes F$ and consider the short exact sequence: $0 \longrightarrow SF \overset{i}{\longrightarrow} A \overset{\pi}{\longrightarrow} E \longrightarrow 0$. In order to prove (i) and (ii), we will use the 6-term exact sequence of K-Theory, see Theorem 1.1.13. From the proof of Theorem 1.1.13 (see [9, Theorem 9.3.1]), we get that $\delta_0 = \alpha - \beta$ and that $\mathrm{K}_1(SF) \simeq \mathrm{K}_0(F)$. Since $\mathrm{K}_0(SF)$ and $\mathrm{K}_1(E)$ are trivial, we have that the following diagram is exact:

$$\begin{array}{ccccc} 0 & \xrightarrow{\mathrm{K}_0(i)} & \mathrm{K}_0(A) & \xrightarrow{\mathrm{K}_0(\pi)} & \mathrm{K}_0(E) \\ {\scriptstyle 0}\uparrow & & & & \downarrow{\scriptstyle \alpha-\beta} \\ 0 & \xleftarrow[\mathrm{K}_1(\pi)]{} & \mathrm{K}_1(A) & \xleftarrow[\mathrm{K}_1(i)]{} & \mathrm{K}_1(SF) \end{array}$$

which gives us the following exact sequence that proves (i)-(ii):

$$0 \longrightarrow \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(E) \overset{\alpha-\beta}{\longrightarrow} \mathrm{K}_0(F) \longrightarrow \mathrm{K}_1(A) \longrightarrow 0$$

(iii) This is a direct application of [3, Theorem 3.1]. □

4.4.5. As the work done in [3] to compute the Cu-semigroup of NCCW 1, we would like to use the pullback structure of (non simple) NCCW 1 complexes to compute their Cu_1-semigroup. However, knowing the explicit computation of $C([0,1])$ and $C(\mathbb{T})$, we deduce that a priori, pullbacks do not pass through Cu_1.
First, observe that the circle algebra can be written as follows: $C(\mathbb{T}) \simeq A(\mathbb{C}, \mathbb{C}, id, id)$. Now consider the pullback (in the category PoM˜):

$$\begin{aligned}\mathrm{Cu}_1(C([0,1])) \underset{\mathrm{Cu}_1(\mathbb{C}\oplus\mathbb{C})}{\oplus} \mathrm{Cu}_1(\mathbb{C}) &:= \{(s,t) \in \mathrm{Cu}_1(C([0,1])) \oplus \mathrm{Cu}_1(\mathbb{C}) \mid (\mathrm{Cu}_1(ev_0), \mathrm{Cu}_1(ev_1))(s) = (t,t)\} \\ &\simeq \{(x,k) \in (\mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \times \mathrm{K}_1(I_x)) \mid x(0) = x(1)\}\end{aligned}$$

$$\mathrm{Cu}_1(C([0,1])) \underset{\mathrm{Cu}_1(\mathbb{C}\oplus\mathbb{C})}{\oplus} \mathrm{Cu}_1(\mathbb{C}) \simeq \mathrm{Cu}_1(C(]0,1[)) \sqcup \mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}_*) \times \{0\}.$$

Define the following PoM˜-morphism:

$$\begin{aligned}\beta : \mathrm{Cu}_1(C(\mathbb{T})) &\longrightarrow \mathrm{Cu}_1(C([0,1])) \underset{\mathrm{Cu}_1(\mathbb{C}\oplus\mathbb{C})}{\oplus} \mathrm{Cu}_1(\mathbb{C}) \\ (x,h) &\longmapsto \begin{cases} (x,h) \text{ if } \operatorname{supp}(x) \neq \mathbb{T} \\ (x,0) \text{ else} \end{cases}\end{aligned}$$

By Theorem 4.3.12, we deduce that β is a surjective morphism. However, since $\beta(1_{|\mathbb{T}}, k) = \beta(1_{|\mathbb{T}}, k')$ for any $k, k' \in \mathbb{Z}$, it is not an order-embedding and a fortiori not an isomorphism. In fact, it is clear that there is no PoM˜-isomorphism between $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}_*) \times \{0\}$ and $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}_*) \times \mathbb{Z}$, since the upper is an upward-directed PoM˜ whereas the latter is not, and hence there is no PoM˜-isomorphism between $\mathrm{Cu}_1(C(\mathbb{T}))$ and $\mathrm{Cu}_1(C([0,1])) \underset{\mathrm{Cu}_1(\mathbb{C}\oplus\mathbb{C})}{\oplus} \mathrm{Cu}_1(\mathbb{C})$.

4.4.6. We conclude this chapter by reminding that Robert ([63]) has been able to classify inductive limits of NCCW 1 by means of an augmented version of the Cu-semigroup. One main restriction of this classification is that the C^*-algebras must have trivial K_1-groups. Even though the Cu_1-semigroup cannot be as nicely computed as the Cu-semigroup, it seems that it would be of interest to investigate on extending the classification of Robert, allowing a torsion-free non trivial K_1 for instance, by means of Cu_1.

Chapter 5

Relation of Cu_1 with existing K-Theoretical invariants

The aim of this chapter is to functorially recover existing invariants and if possible, their classification results. In the first section of this chapter, we explain well-known techniques that have been used over the last decades in order to obtain classification of mainly simple C^*-algebras. Then, in the second section, we functorially recover the K_* group. Finally, we recall which invariant is used to classify AH algebras with the ideal property (see [35]), and check up to which extent we can relate the Cu_1-semigroup to this K-Theoretical invariant.
As before, we shall assume that A is a separable C^*-algebra with has stable rank one. We recall that in order to ease the notations, we use C^* to denote the category of separable C^*-algebras of stable rank one.

5.1 Classification Machinery - Existing work

5.1.1. A first and intuitive approach is to find a functor F that is complete for a certain class of C^*-algebras. That is, any isomorphism between the codomain objects in the codomain category can be lifted to an isomorphism between the C^*-algebras. A second approach, stronger, consists in 'classifying homomorphisms' between C^*-algebras from a 'domain' class to a 'codomain' class. Actually, the latter implies the upper over the intersection of the domain and codomain. One has to make sure that the codomain class is large enough, as one can usually only enlarge the domain subclass. Let us get into details.

Definition 5.1.2. (Complete Invariant)

Let C be a category and let $F : C^* \longrightarrow C$ be a (covariant) functor. We say that F is a *complete invariant* for a certain class of C^*-algebras, C^*_F, if for any A, B in C^*_F such that there exists $F(A) \overset{\alpha}{\simeq} F(B)$, then there exists $A \overset{\phi}{\simeq} B$ such that $F(\phi) = \alpha$. In other words, isomorphisms in C lift to isomorphisms in C^*.

Remark 5.1.3. It can be useful to introduce the notion of *weakly-complete* invariant. This means that an isomorphism at the level of the codomain category implies an isomorphism at the level of C^*-algebras without knowing it actually corresponds to a lift.

Definition 5.1.4. (e.g. [48, Definition 1.10.11])
Let A be a unital C^*-algebra, let $\phi, \psi : A \longrightarrow B$ be *-homomorphisms. We say that ϕ and ψ are *approximately unitarily equivalent*, and we write $\phi \sim_{aue} \psi$, if for every $\epsilon > 0$, and every finite subset E of A, there exists a unitary w of B such that $\|w\phi(x)w^* - \psi(x)\| < \epsilon$, for any $x \in E$.
Equivalently, $\phi \sim_{aue} \psi$ if there exists a sequence of unitary elements $(w_n)_n$ of $B^\sim$ such that $\|w_n\phi(x)w_n^* - \psi(x)\| \underset{n\infty}{\longrightarrow} 0$, for any $x \in A$.

Definition 5.1.5. Let F be a functor from the category C^* into an abstract category C. Let A, B be C^*-algebras. We say that F *classifies homomorphisms* from A to B, if for any $\alpha : F(A) \longrightarrow F(B)$, the two following conditions hold:
(Existence) There exists $\phi : A \longrightarrow B$ such that $F(\phi) = \alpha$.
(Uniqueness) For any other *-homomorphism $\psi : A \longrightarrow B$ such that $F(\psi) = \alpha$, then $\phi \sim_{aue} \psi$.
Those conditions are equivalent with having $(\mathrm{Hom}_{C^*}(A, B))/\sim_{aue} \simeq \mathrm{Hom}_C(F(A), F(B))$.

Remark 5.1.6. Again, it will be useful to consider a functor F that *weakly classifies* homomorphisms, that is, a functor F such that the uniqueness condition holds. Or equivalently, $(\mathrm{Hom}_{C^*}(A, B))/\sim_{aue} \hookrightarrow \mathrm{Hom}_C(F(A), F(B))$.

Lemma 5.1.7. *(Elliott's Intertwining lemma, see e.g. [48, Theorem 1.10.16])*
Let A, B be C^*-algebras. Suppose there exist two *-homomorphisms $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that $\psi \circ \phi \sim_{aue} id_A$ and $\phi \circ \psi \sim_{aue} id_B$. Then, there exists a *-isomorphism $\chi : A \simeq B$ such that $\chi \sim_{aue} \phi$.

5.1.8. This lemma is mostly used in the context of two inductive systems of C^*-algebras that approximately intertwine. Actually, we will obtain an analogous version of this in the category Cu in Chapter 6.

Theorem 5.1.9. *Let C be a category and let $F : C^* \longrightarrow C$ be a functor that classifies homomorphisms from any algebra of C_1^* to any algebra of C_2^*, where C_1^*, C_2^* are subclasses of C^*. Then F is a complete invariant for $C_1^* \cap C_2^*$.*

Proof. Let A, B be in $C_1^* \cap C_2^*$ and let $\alpha : F(A) \simeq F(B)$ be an isomorphism. Then there exist two *-homomorphisms (both unique up to approximate unitary equivalence) $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ such that $F(\phi) = \alpha$ and $F(\psi) = \alpha^{-1}$. Hence, by functoriality of F, we obtain that $F(\psi \circ \phi) = id_{F(A)} = F(id_A)$, and thus $\psi \circ \phi \sim_{aue} id_A$. By a similar argument, we have $\phi \circ \psi \sim_{aue} id_B$. The conclusion follows using Elliott's Intertwining lemma. □

5.1.10. In practice, one has to choose C_2^* to be the largest possible class. However C_1^* can be taken really small (even up to a single algebra), as C_1^* can usually be extended to $\mathcal{A}C_1^* :=$ {Inductive limits of direct sums of building blocks of C_1^* and their unitizations}. As an example, one can look at the classification of AI algebra algebras by means of the Cu-semigroup using the classification of homomorphisms from $C_0(]0,1])$ to any C^*-algebra of stable rank one (see [64]).
We will now define the categorical notion of 'recovering' a functor, its information (and a fortiori, its classification properties), from another one. It allows us to check whether an invariant is 'stronger' than another one.

Definition 5.1.11. Let $C, \mathcal{D}$ be arbitrary categories and let $I : C^* \longrightarrow C$ and $J : C^* \longrightarrow \mathcal{D}$ be (covariant) functors. Let $H : \mathcal{D} \longrightarrow C$ be a functor such that there exists a natural isomorphism $\eta : H \circ J \simeq I$ (see Definition 3.2.28). Then we say we can *recover I from J through H*.

Theorem 5.1.12. *Let $C, \mathcal{D}$ be arbitrary categories and let $I : C^* \longrightarrow C$ and $J : C^* \longrightarrow \mathcal{D}$ be (covariant) functors. Suppose that there exists a functor $H : \mathcal{D} \longrightarrow C$ such that we recover I from J through H.*
(i) If I is a complete invariant for C_I^*, then J is a weakly-complete invariant for C_I^*.
(ii) If I classifies homomorphisms from C_1^* to C_2^*, then J weakly classifies homomorphisms from C_1^* to C_2^*.
If moreover H is faithful, then J is a complete invariant for C_I^* and J classifies homomorphisms from C_1^* to C_2^*. In this case, we say that we can fully recover I from J through H.

Proof. Let I, J and H be functors as in the theorem.
(i) Suppose that I is a complete invariant for C_I^*. Take any two C^*-algebras $A, B \in C_I^*$. If there exists an isomorphism $\alpha : J(A) \simeq J(B)$, by functoriality, we get an isomorphism $H(\alpha)$:

$H \circ J(A) \simeq H \circ J(B)$. Using the natural isomorphism $H \circ J \simeq I$, we know that $H(\alpha)$ gives us an isomorphism $\beta : I(A) \simeq I(B)$. By hypothesis, we can lift β to an isomorphism in the category C^*. That is, there exists a *-isomorphism $\phi : A \simeq B$ such that $I(\phi) = \beta$. We have just shown that J is weakly-complete for C^*_I.
Suppose now that H is faithful. Then the natural isomorphism exactly gives us that $H \circ J(\phi) = H(\alpha)$. Now since H is faithful, we conclude that $J(\phi) = \alpha$. That is, J is a complete invariant for C^*_I.
(ii) Suppose that I classifies homomorphisms from A to B. Let $\alpha : J(A) \longrightarrow J(B)$ be any morphism in $\mathcal{D}$. If $\phi, \psi : A \longrightarrow B$ are *-homomorphisms such that $J(\phi) = J(\psi) = \alpha$, then composing with H, we get $H \circ J(\phi) = H \circ J(\psi) = H(\alpha)$. Thus, $I(\phi) = I(\psi)$, which gives us, by hypothesis, that $\phi \sim_{aue} \psi$. Hence J weakly classifies homomorphisms from A to B.
Finally if H is faithful, then for any $\alpha : J(A) \simeq J(B)$, using again the natural isomorphism $H \circ J \simeq I$, we obtain: For any lift $\phi : A \longrightarrow B$ of $\beta : I(A) \longrightarrow I(B)$, where β is the morphism obtained from $H(\alpha)$ as in the proof of (i) above, we have $H \circ J(\phi) = H(\alpha)$. Since H is faithful, we get that $\alpha = J(\phi)$, from which we deduce that J classifies homomorphisms from A to B. □

5.1.13. We observe that by recovering a functor I from another functor J (through a functor H), we only weakly recover its classification properties. This might seem counter-intuitive, but as J pretends to capture more information on the C^*-algebras, the category considered is usually wider than what we had in the first place. Thus, one usually has to prove again that morphisms indeed lift in a nice way. In practice, H will not be faithful, unless for instance restricting abstractly the category $\mathcal{D}$ of J to ensure that morphisms lift. We will see now concrete uses of this theorem to recover existing classifying functors from Cu_1, and in the process, recall some classification results that have been obtained over the last decades.

Proposition 5.1.14. *By Theorem 3.2.30, we can recover* Cu *and* K_1 *from* Cu_1 *through* ν_+ *and* ν_{max} *respectively. As to be expected, neither* ν_+ *nor* ν_{max} *are faithful functors.*

Proof. We use the natural isomorphisms of Theorem 3.2.30 to directly get the result. □

Corollary 5.1.15. *Let* $\phi, \psi : A \longrightarrow B$ *be two* **-homomorphisms. If* $\mathrm{Cu}_1(\phi) = \mathrm{Cu}_1(\psi)$ *then* $\mathrm{Cu}(\phi) = \mathrm{Cu}(\psi)$ *and* $\mathrm{K}_1(\psi) = \mathrm{K}_1(\phi)$.

5.2 Recovering the K_* invariant

5.2.1. In this section, we give some insight on $\mathrm{K}_* := \mathrm{K}_0 \oplus \mathrm{K}_1$. Although notations might slightly differ, all of this can be found in [25] and [26]. This functor classifies AH_d algebras (a subclass of NCCW 1, see Paragraph 4.4.1) of real rank zero and homomorphisms between $\mathrm{A}\mathbb{T}$ algebras of real rank zero.

5.2.2. *Elliott-Thomsen dimension-drop interval algebras*:
These are one of the first NNCW1 constructed, as they generalize the construction of the circle as a pullback of the interval but in the non-commutative case. Let $q \in \mathbb{N}$. An *Elliott-Thomsen dimension-drop interval* algebra is constructed as $I_q := A(\mathbb{C} \oplus \mathbb{C}, M_q, \pi_0 \otimes 1_q, \pi_1 \otimes 1_q)$, where $\pi_0, \pi_1 : \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C}$ are the respective projections on each component of the direct sum.
By Proposition 4.4.4, we know that:

$$\begin{aligned} \mathrm{K}_0(I_q) &\simeq \mathbb{Z} & \mathrm{Cu}(I_q) &\simeq \{f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \mid f(0), f(1) \in q\overline{\mathbb{N}}\} \\ \mathrm{K}_1(I_q) &\simeq \mathbb{Z}/q\mathbb{Z} & &\simeq \{f \in \mathrm{Lsc}([0,1], \tfrac{1}{q}\overline{\mathbb{N}}) \mid f(0), f(1) \in \overline{\mathbb{N}}\} \end{aligned}$$

5.2.3. An *approximately homogeneous dimensional* algebra, written AH_d algebra, is an inductive limit of finite direct sums of the form $M_n(I_q)$ and $M_n(C(X))$, where $I_q := \{f \in M_q(C([0,1]))$ such that $f(0), f(1) \in \mathbb{C}1_q\}$ and X is one of the following finite connected CW complexes: $\{*\}, \mathbb{T}, [0,1]$. Observe that we have the following inclusions: $\mathrm{AF} \subseteq \mathrm{AI}, \mathrm{A}\mathbb{T} \subseteq \mathrm{AH}_d \subseteq \mathrm{NCCW}\,1$.

Definition 5.2.4. (see e.g. [9, Definition 6.2.1]) An *ordered group* (G, G_+) is an abelian group G together with a distinguished submonoid G_+, called the *positive cone*, such that:
(i) $G_+ - G_+ = G$.
(ii) $G_+ \cap (-G_+) = \{0\}$.
G_+ defines a partial order on G as follows: $x \leq y$ in G if, $y - x \in G_+$. Further, an element $u \in G_+$ is called an *order-unit* if for any $x \in G$ there exists $n \in \mathbb{N}$ such that $x \leq nu$.
We say that the 3-tuple (G, G_+, u) is a *ordered group with order-unit*.

An *ordered group morphism* between two ordered groups $(G, G_+), (H, H_+)$ is a group morphism $\phi : G \longrightarrow H$ such that $\phi(G_+) \subseteq H_+$. If moreover $(G, G_+), (H, H_+)$ admit u, v as respective order-units and if $\phi(u) \leq v$, we say that ϕ *preserves the order-unit*.

Lastly, we define the category of ordered groups with order-unit, denoted AbGp_u, as the category whose objects are ordered groups with order-unit and morphisms are ordered group morphisms that preserve the order-unit.

Lemma 5.2.5. *Recall the Grothendieck construction of Paragraph 1.1.11.*
(i) If S is monoid with cancellation, then $(\mathrm{Gr}(S), S)$ is an ordered group. If moreover S has an order-unit u, then $(\mathrm{Gr}(S), S, u) \in \mathrm{AbGp}_u$.
(ii) Conversely, for any ordered group (G, G_+) such that G_+ is a monoid with cancellation, then $(G, G_+) \simeq (\mathrm{Gr}(G_+), G_+)$ as ordered groups.

Proof. (i) Let S be a monoid with cancellation. We recall that in this case, $\mathrm{Gr}(S) := S \times S / \sim$ where $(m_1, n_1) \sim (m_2, n_2)$ if $m_1 + n_2 = m_2 + n_1$. Naturally define $\mathrm{Gr}(S)_+ := \{[(m, 0)] \in \mathrm{Gr}(S), m \in S\}$. Then $\mathrm{Gr}(S)_+ \simeq S$ and it is easy to see that $\mathrm{Gr}(S)_+ \cap (-\mathrm{Gr}(S)_+) = \{0\}$. Also, observe that $[(m, 0)] + [(0, m)] = [(0, 0)]$ and hence $-\mathrm{Gr}(S)_+ = \{[(0, m)] \in \mathrm{Gr}(S), m \in S\}$, which proves (i).
(ii) Let (G, G_+) be an ordered group and suppose that G_+ has cancellation. We have just proved that $(\mathrm{Gr}(G_+), G_+)$ is an ordered group. Now consider $\alpha : \mathrm{Gr}(G_+) \longrightarrow G$ given by $\alpha([(m, n)]) = m - n$ for any $m, n \in G_+$. It is routine to check that α is a AbGp-isomorphism such that $\alpha(G_+) \subseteq G_+$. □

Definition 5.2.6. Let A be a (unital) C^*-algebra. We define $\mathrm{K}_*(A) := \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$. We also define $\mathrm{K}_*(A)_+ := \{([p]_{\mathrm{K}_0(A)}, [v]_{\mathrm{K}_1(A)})\} \subseteq \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$, where p is a projection in $A \otimes \mathcal{K}$ and v is a unitary in the corner $p(A \otimes \mathcal{K})p$. Notice that we look at the K_1 class of v in A, that is, $[v + (1 - p)]_{\mathrm{K}_1(A)}$. Finally, we define $1_{\mathrm{K}_*(A)} := ([1_A]_{\mathrm{K}_0}, 0_{\mathrm{K}_1})$.

Proposition 5.2.7. *([26, §1.2.2])*
Let $A, B \in C^*_1$. Then $(\mathrm{K}_*(A), \mathrm{K}_*(A)_+)$ is an ordered group and $1_{\mathrm{K}_*(A)} \in \mathrm{K}_*(A)_+$ is an order-unit of $\mathrm{K}_*(A)$. Thus, $(\mathrm{K}_*(A), \mathrm{K}_*(A)_+, 1_{\mathrm{K}_*(A)}) \in \mathrm{AbGp}_u$.
Moreover, for any *-homomorphism $\phi : A \longrightarrow B$, we have $\mathrm{K}_0(\phi) \oplus \mathrm{K}_1(\phi) : \mathrm{K}_*(A) \longrightarrow \mathrm{K}_*(B)$ is an ordered group morphism that preserves the order-unit. We write $\mathrm{K}_*(\phi) := \mathrm{K}_0(\phi) \oplus \mathrm{K}_1(\phi)$.
Thus, we obtain a covariant functor:

$$\begin{aligned} \mathrm{K}_* : AH_d &\longrightarrow \mathrm{AbGp}_u \\ A &\longmapsto (\mathrm{K}_*(A), \mathrm{K}_*(A)_+, 1_{\mathrm{K}_*(A)}) \\ \phi &\longmapsto \mathrm{K}_*(\phi) \end{aligned}$$

5.2.8. We do not give a proof of the above, but we remind the reader that whenever a C^*-algebra A has stable rank one -which is the case of any AH_d algebra-, then the monoid $V(A)$

has cancellation and hence $\mathrm{K}_0(A)_+$ can be identified with $V(A)$ and thus $(\mathrm{K}_0(A), V(A))$ is an ordered group.
Also, for any *-homomorphism $\phi : A \longrightarrow B$, since that $\phi_{|pAp} : pAp \longrightarrow qBq$, where $q := \alpha(p)$, one can check that $\mathrm{K}_*(\phi)$ is indeed a well-defined ordered group homomorphism that respects the order-unit.
Finally, we also recall that in the stable rank one case, the Murray-von Neumann equivalence and the Cuntz equivalence agree on the projections of $A \otimes \mathcal{K}$ and that $V(A) \simeq \mathrm{Cu}(A)_c$. That is, any compact element of $\mathrm{Cu}(A)$ is the class of some projection of $A \otimes \mathcal{K}$.
Among classification results obtained in the references, we will only mention two notable ones that catch our interest:

Theorem 5.2.9. *([26, Corollary 4.9], [25, Theorem 7.3 - Theorem 7.4])*
(i) The functor K_* *is a complete invariant for (unital)* AH_d *algebras of real rank zero.*
(ii) Let A, B *be (unital)* $\mathrm{A}\mathbb{T}$ *algebras of real rank zero and let* $\alpha : \mathrm{K}_*(A) \longrightarrow \mathrm{K}_*(B)$ *be a scaled ordered group morphism. Then there exists a unique* **-homomorphism (up to approximate unitary equivalence)* $\phi : A \longrightarrow B$ *such that* $\mathrm{K}_*(\phi) = \alpha$.

5.2.10. The aim now is to recover K_* from Cu_1 and thus show that Cu_1 contains more information than K_*. For that purpose, we first need to define the category of $\mathrm{Cu}^\sim$-semigroups with order-unit, that will be written $\mathrm{Cu}^\sim_u$. Further, we are going to create a functor $H_* : \mathrm{Cu}^\sim_u \longrightarrow \mathrm{AbGp}_u$ such that $H_* \circ \mathrm{Cu}_1 \simeq \mathrm{K}_*$ as functors. Moreover, restricting to an adequate subcategory of $\mathrm{Cu}^\sim_u$, we will see that H_* is faithful.

Definition 5.2.11. Let S be a $\mathrm{Cu}^\sim$-semigroup. We say that S has *weak cancellation* if $x + z \ll y + z$ implies $x \leq y$ for $x, y, z \in S$. We say that S has *cancellation of compact elements* if $x + z \leq y + z$ implies $x \leq y$ for any $x, y \in S$ and $z \in S_c$.

Lemma 5.2.12. *Let* S *be a* $\mathrm{Cu}^\sim$*-semigroup. If* S *has weak cancellation, then* S *has cancellation of compact elements.*

Proof. Let $S \in \mathrm{Cu}^\sim$ such that S has weak cancellation. Let $x, y \in S$ and let $z \in S_c$. Suppose that $x + z \leq y + z$. Consider a $\ll$-increasing sequence $(x_n)_n$ in S whose supremum is x given by (O2). Since $z \ll z$ and since $x_n \ll y$ for any n, we have that $x_n + z \ll y + z$ for any $n \in \mathbb{N}$. Using weak cancellation and passing to suprema, we deduce that $x \leq y$, which ends the proof. □

Proposition 5.2.13. *([63, Proposition 2.1.3],[68, Proposition 4.2 - Theorem 4.3])*
Let $A \in C^*$. *Then* $\mathrm{Cu}(A)$ *has weak cancellation and a fortiori* $\mathrm{Cu}(A)$ *has cancellation of compact elements.*

Corollary 5.2.14. *Let $A \in C^*$. Then $\mathrm{Cu}_1(A)$ has weak cancellation and a fortiori $\mathrm{Cu}_1(A)$ has cancellation of compact elements.*

Proof. Combine Proposition 5.2.13 with Corollary 2.1.15 to get the result. □

Definition 5.2.15. Let S be a countably-based positively directed and positively convex $\mathrm{Cu}^\sim$-semigroup. Suppose that S has cancellation of compact elements. Also suppose that S_+ admits a compact order-unit. That is, there exists $u \in S_{+,c}$ such that u is an order-unit of S_+ (see Definition 1.3.8).
We say that (S, u) is a $\mathrm{Cu}^\sim$*-semigroup with compact order-unit*. Now, a $\mathrm{Cu}^\sim$*-morphism* between two $\mathrm{Cu}^\sim$-semigroups with compact order-unit $(S, u), (T, v)$ is a $\mathrm{Cu}^\sim$-morphism $\alpha : S \longrightarrow T$ such that $\alpha(u) \leq v$.
We define the category of $\mathrm{Cu}^\sim$-semigroups with compact order-unit, denoted $\mathrm{Cu}^\sim_u$, as the category whose objects are $\mathrm{Cu}^\sim$-semigroups with order-unit and morphisms are $\mathrm{Cu}^\sim$-morphisms that preserve the order-unit.

Lemma 5.2.16. *The assignment*

$$\begin{aligned} \mathrm{Cu}_{1,u} : C^*_1 &\longrightarrow \mathrm{Cu}^\sim_u \\ A &\longmapsto (\mathrm{Cu}_1(A), ([1_A], 0)) \\ \phi &\longmapsto \mathrm{Cu}_1(\phi) \end{aligned}$$

from the category of unital separable C^*-algebras of stable rank one, denoted by C^*_1, to the category $\mathrm{Cu}^\sim_u$ is a covariant functor.

Proof. As stated in Proposition 5.2.13, we know that $\mathrm{Cu}_1(A)_+$ has cancellation of compact elements. Plus, we know that $([1_A], 0)$ is a compact order-unit of $\mathrm{Cu}_1(A)_+$, so it easily follows that $\mathrm{Cu}_{1,u}(A) \in \mathrm{Cu}^\sim_u$. Finally, it is trivial to see that $\mathrm{Cu}_1(\phi)([1_A]) \leq [1_B]$, which ends the proof. □

Lemma 5.2.17. *The assignment*

$$\begin{aligned} H_* : \mathrm{Cu}^\sim_u &\longrightarrow \mathrm{AbGp}_u \\ (S, u) &\longmapsto (\mathrm{Gr}(S_c), S_c, u) \\ \alpha &\longmapsto \mathrm{Gr}(\alpha_c) \end{aligned}$$

from the category $\mathrm{Cu}^\sim_u$ to the category AbGp_u is a covariant functor.
Moreover, if we restrict the domain of H_* to the category of algebraic $\mathrm{Cu}^\sim_u$-semigroups with compact order-unit, denoted by $\mathrm{Cu}^\sim_{u,alg}$, then H_* becomes a faithful functor.

Proof. Let $(S,u) \in \mathrm{Cu}_u^\sim$. By Corollary 5.2.14, we know that S_c is a monoid with cancellation and hence, using Lemma 5.2.5, we deduce that $(\mathrm{Gr}(S_c), S_c, u)$ is an ordered group with order-unit. Now let $\alpha : S \longrightarrow T$ be a $\mathrm{Cu}_u^\sim$-morphism between two $\mathrm{Cu}^\sim$-semigroups with order-unit $(S,u),(T,v)$. By functoriality of ν_c, it follows that $\alpha_c : S_c \longrightarrow T_c$ is a $\mathrm{PoM}^\sim$-morphism, and hence that $\mathrm{Gr}(\alpha_c) : \mathrm{Gr}(S_c) \longrightarrow \mathrm{Gr}(T_c)$ is a group morphism such that $\mathrm{Gr}(\alpha_c)(S_c) \subseteq T_c$. Finally, using that $\alpha(u) \leq \alpha(v)$, we obtain $\mathrm{Gr}(\alpha_c)(u) \leq v$. We conclude that H_* is a well-defined functor.
Now, we have to show that if we restrict the domain of H_* to $\mathrm{Cu}_{sc,alg}^\sim$, then H_* becomes faithful. Let $\alpha, \beta : (S,u) \longrightarrow (T,v)$ be two scaled $\mathrm{Cu}^\sim$-morphisms between $(S,u),(T,v) \in \mathrm{Cu}_{sc,alg}^\sim$ such that $H_*(\alpha) = H_*(\beta)$. In particular, $\alpha_c = \beta_c$, and since we are in the category of algebraic $\mathrm{Cu}^\sim$-semigroups, any element is supremum of increasing sequences of compact elements. Thus any morphism is entirely determined by its restriction to compact elements. One can conclude that $\alpha = \beta$ which terminates the proof. □

Theorem 5.2.18. *The functor* $H_* : \mathrm{Cu}_u^\sim \longrightarrow \mathrm{AbGp}_u$ *in Lemma 5.2.17 yields a natural isomorphism* $\eta_* : H_* \circ \mathrm{Cu}_{1,u} \simeq \mathrm{K}_*$.

Proof. First we prove that $\mathrm{K}_*(A)_+ \simeq \mathrm{Cu}_1(A)_c$ as monoids and the result will follow from Lemma 5.2.5 (ii).
By Proposition 2.4.4 we know that $\mathrm{Cu}_1(A)_c$ is a monoid. Now consider $[(a,u)] \in \mathrm{Cu}_1(A)_c$. By Corollary 2.1.15, we know that $[a]$ is a compact element of $\mathrm{Cu}(A)$. Besides, since A has stable rank one, we know that we can find a projection $p \in A \otimes \mathcal{K}$ such that $[p] = [a]$ in $\mathrm{Cu}(A)$. So without loss of generality, we now describe compact elements of $\mathrm{Cu}_1(A)$ as classes $[(p,u)]$ where p is projection in $A \otimes \mathcal{K}$ and u is a unitary element in her p.
On the other hand, by Theorem 3.2.30, we have $\mathrm{Cu}_1(A)_{max} \simeq \mathrm{K}_1(A)$, where the AbGp-isomorphism is given by $[(s_{A\otimes\mathcal{K}}, u)] \longmapsto [u]$, where $s_{A\otimes\mathcal{K}}$ is any strictly positive element of $A \otimes \mathcal{K}$. Combined with Proposition 3.3.14, we get a monoid morphism $j : \mathrm{Cu}_1(A) \longrightarrow \mathrm{K}_1(A)$. Now set:

$$\begin{aligned} \alpha : \mathrm{Cu}_1(A)_c &\longrightarrow \mathrm{K}_*(A)_+ \\ [(p,u)] &\longmapsto ([p], j([p,u])) \end{aligned}$$

It is routine to check that α is monoid morphism. Further, observe that $j([p,u]) = \delta_{I_pA}([u])$ for any $[(p,u)] \in \mathrm{Cu}_1(A)_c$, where $\delta_{I_pA} : \mathrm{K}_1(\text{her } p) \overset{\mathrm{K}_1(i)}{\longrightarrow} \mathrm{K}_1(A)$ (see Definition 3.1.7). Thus, $j([p,u]) = [u + (1-p)]_{\mathrm{K}_1(A)}$. Furthermore, since A has stable rank one, the Murray-von Neumann equivalence and the Cuntz equivalence agree on projections. It is now clear that α is an isomorphism and hence $\mathrm{Cu}_1(A)_c \simeq \mathrm{K}_*(A)_+$ as monoids. Using Lemma 5.2.5 (ii),

it follows that $(K_*(A), K_*(A)_+) \simeq (\mathrm{Gr}(\mathrm{Cu}_1(A)_c), \mathrm{Cu}_1(A)_c)$ as ordered groups. Finally, it is routine to check that $[(1_A, 1_A)]$ is a compact order-unit for $\mathrm{Cu}_1(A)$ (a fortiori, an order-unit for $(\mathrm{Gr}(\mathrm{Cu}_1(A)_c), \mathrm{Cu}_1(A)_c)$) and that $\alpha([(1_A, 1_A)]) = 1_{K_*(A)}$.
We conclude that for any $A \in C_1^*$, there exists a natural ordered group isomorphism η_{*A} : $H_* \circ \mathrm{Cu}_{1,u}(A) \simeq (K_*(A), K_*(A)_+, 1_{K_*(A)})$ that preserves the order-unit and hence there exists a natural isomorphism $\eta_* : H_* \circ \mathrm{Cu}_{1,u} \simeq K_*$. □

Corollary 5.2.19. *By restricting to the category* $\mathrm{Cu}^{\sim}_{u,alg}$*, we can fully recover* K_* *from* $\mathrm{Cu}_{1,u}$ *through* H_**. A fortiori, we have:*
(i) $\mathrm{Cu}_{1,u}$ *is a complete invariant for* AH_d *algebras of real rank zero.*
(ii) $\mathrm{Cu}_{1,u}$ *classifies homomorphisms of* $A\mathbb{T}$ *algebras with real rank zero.*

5.2.20. (Open line of research)
As briefly explained in the introduction, the original Elliott invariant has been extended over the years, on the one hand to reajust the conjecture of the classification program when needed and on the other hand to be able to classify other classes of C^*-algebras; in particular in the non-simple case.
In the work of Gong, Jiang, Li and Pasnicu (among others, -see [34], [35]-) a new notion and a new invariant to classify a rather large class of AH algebra containing the simple and the real rank case, was introduced: AH algebras with ideal property of no dimension growth. The new invariant created was called $\mathrm{Inv}^0(A) := (\underline{K}(A), \underline{K}(A)_+, \Sigma(A), \{\mathrm{Aff}\, T(pAp)\}_{[p]\in\Sigma(A)})$, where $\underline{K}$ is the total K-Theory, $\Sigma(A)$ is scale of $K_0(A)$, and $\mathrm{Aff}\, T(pAp)$ is the Banach space of continuous affine maps from the tracial space of corner algebras to $\mathbb{R}$, together with compatibility conditions. Indeed, they finally manage to classify a subclass of well-behaved AH algebras with the ideal property of no dimension growth by means of Inv^0, to subsequently classify all AH algebras with the ideal property and no dimension growth adding a new ingredient to Inv^0 that involves the Hausdorffized K_1 group of corner algebras and some more compatibility axioms, see [34], [35].
We have been investigating whether it would be possible to recover this kind of information, at least partially. To do so, we would have to create a suitable category that would allow us to abstractly describe the objects and the compatibility axioms that define these invariants. It would seem that we are able to recover $\{\mathrm{Aff}\, T(pAp)\}_{[p]\in\Sigma(A)}$ and its compatibility axioms, together with the Hausdorffized K_1 group of corner algebras, whereas the compatibility axioms for the latter and the total K-Theory seem more problematic.

Chapter 6

Intertwinings in the category Cu

This chapter contains tools and techniques that will be defined for Cu-semigroups of the form $\mathrm{Lsc}(X, S)$, where X is a compact Hausdorff metric space of covering dimension 1 and S is a countably-based Cu-semigroup. In the first section, we define the notion n-piecewise characteristic functions, describe their properties, and prove that the set of these lower-semicontinuous maps is in fact a countable basis for $\mathrm{Lsc}(X, S)$. The second section is focused on metrics in the category Cu. The third and fourth section are analogous versions of approximate intertwinings in the category Cu.
Observe that all of this will be of a particular interest in a next chapter, as it will allow us to use an approximate intertwining argument in a context linked to the Cuntz semigroup of NCCW 1 algebras.

6.1 Piecewise characteristic functions

Definition 6.1.1. [28, Definition 1.1.1]
Let X be a topological space and let $\mathcal{U} = \{U_k\}_{k\in\Lambda}$ be a cover of X. For any $x \in X$, we call the *multiplicity* of x in $\mathcal{U}$, the number of sets of $\mathcal{U}$ containing x. Furthermore, we say that X is of *covering dimension n*, if for any open cover $\mathcal{U}$ of X, there exists an open refinement of $\mathcal{U}$ of multiplicity at most $n + 1$, that is, any $x \in X$ belongs to at most $n + 1$ open sets of the refinement.

6.1.2. In the entire chapter, X is a compact metric space (hence Hausdorff, second countable) of covering dimension 1 and S is a countably-abased Cu-semigroup.

Definition 6.1.3. Let X be a topological metric space of covering dimension 1. We call a *1-thin cover* of X, any closed finite cover $\mathcal{U} := \{\overline{U_k}\}_{k=1}^m$ of X where any U_k is an open connected set of X and such that for any $k \neq l$, we have $U_k \cap U_l = \emptyset$.
Furthermore, if there exist m points $\{y_k\}_{k=1}^m$ of X and a constant $R \in \mathbb{R}_+$ such that for any k, we have $U_k = B(y_k, R/m)$, then we say that $\mathcal{U}$ is a *1-thin cover of size* $1/m$.

Proposition 6.1.4. *Any compact (Hausdorff) metric space X of covering dimension* 1 *admits 1-thin covers.*

Proof. Let X be a compact (Hausdorff) metric space of covering dimension 1. Let $\mathcal{V} := \{V_k\}_{k\in\Lambda}$ be an open cover of X. We can find a refinement $\mathcal{W}$ of $\mathcal{V}$ that has multiplicity 1. Besides, since X is a metric space, we can suppose that $\mathcal{W} := \{W_k\}_{k\in\Lambda}$ are open balls of X. Using compactness of X, we can extract a finite cover $\mathcal{W}' := \{W_k\}_{k=1}^n$ of $\mathcal{W}$. Thus, we have constructed a finite cover of open balls of X that has multiplicity 1. Now let us denote the interior of a set X by X° and let us consider:

$$V_k := (W_k \setminus (\underset{l\neq k}{\cup} W_l))^\circ \qquad\qquad V_{ll'} := W_l \cap W_{l'}$$

for any $k, l, l'\{1, ..., n\}$ such that $l \neq l'$.
Write $\mathcal{U} := \{\overline{V_k}\}_k \cup \{\overline{V_{ll'}}\}_{l\neq l'}$. We will prove that $\mathcal{U}$ is a 1-thin cover of X.
Let $k, l, l'\{1, ..., n\}$ such that $l \neq l'$. Observe that $W_k \setminus (\underset{l\neq k}{\cup} W_l) = \underset{l\neq k}{\cup}(W_k \setminus (W_k \cap W_l))$. It follows that $V_k \cup V_{ll'} = \emptyset$. Also, it is easy to see that $V_k, V_{ll'}$ are open connected sets. We only have to check that $\mathcal{U}$ is a closed cover of X.
Let $x \in X$. Since $\mathcal{W}'$ is a finite cover of X, there exists $k \in \{1, ..., n\}$ such that $x \in W_k$. From the observation made just above, one can deduce that $\overline{W_k} = (\underset{l\neq k}{\cup} \overline{V_{kl}}) \cup \overline{V_k}$. We conclude that there exists a closed set of $\mathcal{U}$ containing x and hence the result follows. □

Definition 6.1.5. Let X be compact metric space of covering dimension 1 and let S be a countably-based Cu-semigroup. Consider the Cu-semigroup $\mathrm{Lsc}(X, S)$. A map $g : X \longrightarrow S$, is called a *piecewise characteristic function* of $\mathrm{Lsc}(X, S)$ if there exists a 1-thin cover $\{\overline{U_k}\}_{k=1}^m$ of X, and m elements $s_1, ..., s_m$ of S such that $g_{|U_k} = s_k$, for any $1 \leq k \leq m$ and such that $g \in \mathrm{Lsc}(X, S)$. The set of piecewise characteristic functions of $\mathrm{Lsc}(X, S)$ will be denoted by $\chi(X, S)$.
If moreover $g \ll f$ for some $f \in \mathrm{Lsc}(X, S)$, we say g is an piecewise characteristic function for f. The set of piecewise characteristic functions of $\mathrm{Lsc}(X, S)$ for f will be denoted by $\chi_f(X, S)$.

6.1.6. When clear, we will omit X and S in the notation $\chi(X, S)$. Also, this definition has been adapted from [3, Definition 2.4 - Definition 5.9] to fit our setting, but one can check that the following proposition still applies.

Proposition 6.1.7. *[3, Proposition 5.13 - Proposition 5.14]*
(i) χ_f is upward-directed. That is, for any g_1, g_2 in χ_f, there exists $g \in \chi_f$ such that $g_1, g_2 \ll g$.
(ii) For any $f \in \mathrm{Lsc}(X, S)$, we have $f = \sup\{g, g \in \chi_f\}$. Actually, one can find a $\ll$-increasing sequence $(g_l)_l$ in χ_f such that $f = \sup\limits_{l \in \mathbb{N}} g_l$.

6.1.8. We will now focus on the context we will be working on: the interval and the circle. So from now on, X is either the circle or the interval.
In fact, we mention that it might be possible to generalize the following to any topological space X that admits 1-thin covers of some size. Also, we conjecture that any connected compact (Hausdorff) metric space X of covering dimension 1 admits 1-thin covers of some size, or equivalently, of any size.
Let us now construct *the canonical 1-thin covers* of the circle and the interval, of any size:

6.1.9. Let $X := [0, 1]$. Take an equidistant partition of the interval $x_0 = 0, x_1, ..., x_{m-1}, x_m = 1$, that is, $x_k := k/m$. Consider $U_1 := [0, x_1[$, $U_m :=]x_m, 1]$ and for any $2 \leq k \leq m-1$, $U_k :=]x_{k-1}, x_k[$. Then, $\{\overline{U_k}\}_{k=1}^m$ is a 1-thin cover of $[0, 1]$ of size $1/m$.

Let $X := \mathbb{T}$. Take an equidistant partition of the circle $x_0 = 1, x_1, ..., x_{m-1}, x_m = x_0$ starting at 1. That is, $x_k := e^{2ik\pi/m}$. Consider $U_k :=]x_{k-1}, x_k[$.. Then $\{\overline{U_k}\}_{k=1}^m$ is a 1-thin cover of $\mathbb{T}$ of size $1/m$. Observe that any 1-thin cover of $\mathbb{T}$ of size $1/m$ is of this form. In fact, they are entirely determined by their initial point x_0.

The explicit 1-thin covers constructed above will be referred to as *the canonical 1-thin covers of X of size* $1/m$.

Definition 6.1.10. Let X be either the circle or the interval and let S be a countably-based Cu-semigroup. Let $g \in \mathrm{Lsc}(X, S)$ such that $g \ll \infty$. We say that g is an *n-piecewise characteristic function* of $\mathrm{Lsc}(X, S)$ if there exist a size $n \in \mathbb{N}$ and 3^n elements $s_1, ..., s_{3^n}$ of S such that $g_{|U_k} = s_k$ for any $1 \leq k \leq 3^n$, where $\{\overline{U_k}\}_{k=1}^{3^n}$ is the canonical 1-thin cover of X of size $1/3^n$.
The set of n-piecewise characteristic functions of $\mathrm{Lsc}(X, S)$ will be denoted by $\chi_n(X, S)$.
If moreover $g \ll f$ for some $f \in \mathrm{Lsc}(X, S)$, we say g is an n-piecewise characteristic function for f. The set of n-piecewise characteristic functions of $\mathrm{Lsc}(X, S)$ for f will be denoted by

$\chi_{n,f}(X,S)$.
Finally, in the case that $S=\overline{\mathbb{N}}$, we define $\Gamma_n(X)$ (or Γ_n when the context is clear) as the finite subset of $\chi_n(X,\overline{\mathbb{N}})$ consisting of n-piecewise characteristic functions taking values in $\{0,1\}$.

Proposition 6.1.11. *Let X be either the circle or the interval and let S be a countably-based* Cu*-semigroup.*
(i) For any $n\in\mathbb{N}$, we have $\chi_n(X,S)\subset\chi(X,S)$, (respectively, $\chi_{n,f}\subset\chi_f$).
(ii) Whenever $n<m$, we have $\chi_n(X,S)\subset\chi_m(X,S)$ (respectively $\chi_{n,f}\subset\chi_{m,f}$). In particular, any n-piecewise characteristic function is also an $(n+1)$-piecewise characteristic function.
(iii) For any two $g,g'\in\chi_n$, we have $g+g'\in\chi_n$.

Proof. (i) is clear from the definitions.
(ii) Let $n<m\in\mathbb{N}$ and let $\mathcal{U}:=\{\overline{U_k}\}_1^{3^n},\mathcal{W}:=\{\overline{W_i}\}_1^{3^m}$ be the canonical 1-thin covers of X of size $1/3^n,1/3^m$ respectively. Let $g\in\chi_n$. Observe that for any $1\leq k\leq 3^n$, we can find 3^{m-n} open sets $W_{i,k}$ of $\mathcal{W}$ such that $W_{i,k}\subseteq U_k$ and such that $\{\overline{W_{i,k}}\}_{i=1}^{3^{m-n}}$ is a 1-thin cover of $\overline{U_k}$ of size $1/3^{m-n}$. Plus, we know that there exists $s_1,...,s_{3^n}\in S$ such that $g_{|U_k}=s_k$ for any $k\in\{1,...,3^n\}$ and $g\in\mathrm{Lsc}(X,S)$. Now define $h:X\longrightarrow S$ as follows:
(1) For any $1\leq k\leq 3^n$, put $h_{|W_{1,k}}=h_{|W_{3,k}}=h_{|\overline{W_{2,k}}}=s_k$.
(2) For any $x\in X\setminus(\bigcup\limits_{k=1}^{3^n}U_k)$, put $h(x)=g(x)$.
By construction, we have $h\in\chi_m$ and also that $g=h$. Thus $\chi_n\subseteq\chi_m$.
(iii) Let $g,g'\in\chi_n$. We know that there exist $s_1,...,s_{3^n}\in S$ (respectively $s'_1,...,s'_{3^n}\in S$) such that $g_{|U_k}=s_k$ and $g\in\mathrm{Lsc}(X,S)$ (similarly for h and $s'_1,...,s'_{3^n}$). Then $(g+g')_{|U_k}=s_k+s'_k$ and $g+g'\in\mathrm{Lsc}(X,S)$ since $\mathrm{Lsc}(X,S)$ is closed under point-wise addition. Finally, since $g,g'\ll\infty$, we also have $g+g'\ll\infty$. So $g+g'\in\chi_n$, which ends the proof. □

Corollary 6.1.12. *Let X be either the circle or the interval and let S be a countably-based* Cu*-semigroup. It follows that χ_n and $\bigcup\limits_{n\in\mathbb{N}}\chi_n$ are* PoM.

Proof. Using Proposition 6.1.11 (iii), we know that each χ_n is stable under addition and since addition and order are compatible in $\mathrm{Lsc}(X,S)$, a fortiori they are compatible in χ_n. It follows that χ_n is a PoM. Now take two elements $g,g'\in\bigcup\limits_{n\in\mathbb{N}}\chi_n$. Then there exists l,l' such that $g\in\chi_l$ and $g'\in\chi_{l'}$. From Proposition 6.1.11 (ii), we know that in fact both belong to a common χ_n. Indeed we can take any $n\geq\max(l,l')$. We conclude that $g+g'\in\chi_n$ and again addition and order are compatible. □

6.1.13. Notations: Let X be a compact metric space of covering dimension 1 and consider the Cu-semigroup $\mathrm{Lsc}(X,\overline{\mathbb{N}})$. Observe that the open sets of X, that we write $\mathcal{O}(X)$ (see Corollary 4.3.5), are in canonical bijection with $\{1_U\}_{U\subseteq X}$. So we might interchangeably use one object or the other. In fact, we have $1_U \leq 1_V$ if and only if $U \subseteq V$. We sometimes write $U \leq V$. Also, $1_U \ll 1_V$ if and only if there exists a compact K of X such that $U \subseteq K \subseteq V$. We sometimes write $U \ll V$.

Lemma 6.1.14. *Let X be a compact metric space of covering dimension 1 and consider the* Cu*-semigroup* $\mathrm{Lsc}(X,\overline{\mathbb{N}})$*. Any $f \in \mathrm{Lsc}(X,\overline{\mathbb{N}})$ can be uniquely described by a $\subseteq$-decreasing sequence of open sets in X. Indeed, $f = \sum\limits_{n=0}^{\infty} 1_{U_n}$, where $U_n := f^{-1}(]n;+\infty])$.*
For any $f, g \in \mathrm{Lsc}(X,\overline{\mathbb{N}})$, write $(U_n)_n$ and $(V_n)_n$ the decreasing sequences of open sets that uniquely determine f and g. That is, $f = \sum\limits_{n=0}^{\infty} 1_{U_n}$ and $g = \sum\limits_{n=0}^{\infty} 1_{V_n}$. Then:
(i) $f \leq g$ if and only if $U_n \leq V_n$ for any $n \in \mathbb{N}$.
(ii) If moreover $f \ll \infty$, then $f \ll g$ if and only if $U_n \ll V_n$ for any $n \in \mathbb{N}$.
We refer to both the decreasing sequence $(U_n)_n$ that uniquely determine an element $f \in$ $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ *and the countable sum $\sum\limits_{n=0}^{\infty} 1_{U_n}$ as the* canonical decomposition of f.

Proof. From the lower semicontinuity of f, we know that $U_n := f^{-1}(]n;+\infty])$ is an open set of X. Clearly, $(U_n)_n$ is a decreasing sequence in $\mathcal{O}(X)$ and $f = \sum\limits_{n=0}^{\infty} 1_{U_n}$. Now let $f, g \in \mathrm{Lsc}(X,\overline{\mathbb{N}})$ and $(U_n)_n, (V_n)_n$ be their respective decreasing sequences.
(i) If $U_n \leq V_n$ for any $n \in \mathbb{N}$, then by the Cuntz axioms (O3) and (O4), we deduce that $f \leq g$. Conversely, if $f \leq g$, then for any $t \in X$, we have $f(t) \leq g(t)$. Observe that in fact, $f(t) = \#\{n : t \in U_n\}$, hence we deduce that $U_n \leq V_n$ for any $n \in \mathbb{N}$.
It follows that the decreasing sequence of open sets that determines f (respectively g) is unique.
(ii) If $U_n \ll V_n$ for any $n \in \mathbb{N}$, then by the Cuntz axioms (O3) and (O4), we deduce that $f \ll g$. Conversely, if $f \ll g$, then for any $n \in \mathbb{N}$, using (O2), take a $\ll$-increasing sequence $(V_{n,i})_{i\in\mathbb{N}}$ whose supremum is V_n. It follows (from what we have just proved) that $(\sum\limits_{n=0}^{\infty} 1_{V_{n,i}})_i$ is a $\ll$-increasing sequence whose supremum is g. Hence, we can find some $i \in \mathbb{N}$ such that $f \leq \sum\limits_{n=0}^{\infty} 1_{V_{n,i}}$. That is, using (i), $U_n \leq V_{n,i} \ll V_n$, for all $n \in \mathbb{N}$, which ends the proof. □

Lemma 6.1.15. *Let X be the circle or the interval and consider the* Cu*-semigroup* $\mathrm{Lsc}(X,\overline{\mathbb{N}})$.
Let $l \in \mathbb{N}$. Then:
(i) Let $g \in \chi_l$. Then, its canonical decomposition $\sum\limits_{n=0}^{\infty} 1_{V_n}$ is a finite sum of elements of Γ_l.
(ii) Let $g, h \in \Gamma_l$. Then the canonical decomposition of $g + h \in \chi_l$ is $g + h = 1_{\mathrm{supp}\, g \cap \mathrm{supp}\, h} + 1_{\mathrm{supp}\, g \cup \mathrm{supp}\, h}$. A fortiori, $g + h \in \Gamma_n$ if and only if $\mathrm{supp}\, g \cap \mathrm{supp}\, h = \emptyset$.

(iii) Let $n < m$. For any two $g, g'' \in \chi_n$ such that $g'' \ll g$, we can find $g' \in \chi_m$ such that $g'' \ll g' \ll g$ in χ_m.

Proof. Consider $\{U_k\}_{k=1}^{3^l}$ the canonical 1-thin cover of X of size $1/3^l$.
(i) Let $g \in \chi_l$. Since $g \ll \infty$, we know that $g(t) \ll \infty$ for any $t \in X$. Thus, there exists $n \in \mathbb{N}$ such that $g(t) \leq n$ for any $t \in \mathbb{N}$. We recall that $V_n := g^{-1}(]n; +\infty])$. We deduce that $V_{n+1} = V_m = \emptyset$ for any $m \geq n+1$. That is, the canonical decomposition of g is a finite sum. Furthermore, since g is constant on U_k, then so is 1_{V_i}, for any $i \leq n$. That is, 1_{V_i} is an element of Γ_l, for any $i \leq n$ and we obtain (i).
(ii) For any $g, h \in \mathrm{Lsc}(X, \{0, 1\})$, we have that $g + h \in \mathrm{Lsc}(X, \{0, 2\})$. Moreover, the canonical decomposition of $g + h$ is $1_{\mathrm{supp}\, g \cap \mathrm{supp}\, h} + 1_{\mathrm{supp}\, g \cup \mathrm{supp}\, h}$. Now consider $g, h \in \Gamma_l$. By (i), we know that $1_{\mathrm{supp}\, g \cup \mathrm{supp}\, h}, 1_{\mathrm{supp}\, g \cap \mathrm{supp}\, h} \in \Gamma_n$, from which we deduce (ii).
(iii) Let $n \in \mathbb{N}$ and let $\mathcal{U} := \{\overline{U_k}\}_1^{3^n}$ be the canonical 1-thin cover of X of size $1/3^n$. Let $g, g'' \in \chi_n$ such that $g'' \ll g$. From (i), we know that both of their canonical decomposition are finite sums of elements of Γ_n. Plus, by Lemma 6.1.14 (iii), we know that $g'' \ll g$ if and only if $1_{V_k''} \ll 1_{V_k}$ in Γ_n, for any k, where $\sum\limits_k 1_{V_k}, \sum\limits_k 1_{V_k''}$ are the canonical decompositions of g, g'' respectively. So without loss of generality, we suppose that $g, g'' \in \Gamma_n$.
Write $V'' := \mathrm{supp}\, g''$ and $V := \mathrm{supp}\, g$. Both V, V'' have a finite number of (open) connected components. First, suppose that V'', V are open connected sets. We know that $V'' \ll V$ and also that $g'', g \in \Gamma_n$. Thus, we have $V := U_r \cup (\bigcup\limits_{k=r+1}^{s-1} \overline{U_k}) \cup U_s$, $V'' := U_{r''} \cup (\bigcup\limits_{k=r''+1}^{s''-1} \overline{U_k}) \cup U_{s''}$, for some $r' \leq r \leq s \leq s'$. Since $V \ll V'$, either $V = V' = X$ or else $r' < r \leq s < s'$. The first case being trivial, we suppose now that $r' < r \leq s < s'$.
Now let $m \in \mathbb{N}$ such that $n < m$ and let $\mathcal{W} := \{\overline{W_i}\}_1^{3^m}$ be the canonical 1-thin cover of X of size $1/3^m$. Observe that for any $1 \leq k \leq 3^n$, we can find 3^{m-n} open sets $W_{i,k}$ of $\mathcal{W}$ such that $W_{i,k} \subseteq U_k$ and such that $\{\overline{W_{i,k}}\}_{i=1}^{3^{m-n}}$ is a 1-thin cover of $\overline{U_k}$ of size $1/3^{m-n}$. Now, define $g' := 1_{V'}$, where $V' := W_{3^{m-n}, r''-1} \cup \overline{V''} \cup W_{1, s''+1}$. By construction, $g' \in \chi_m$ and $g'' \ll g' \ll g$.
Finally, as mentioned above, both V, V'' have a finite number of (open) connected components so it is easy to check that the result follows. □

Remark 6.1.16. We will often say that $\{1_U\}_{U \in O(X)}$ *generates* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. Abusing the language, we also say that Γ_n generates χ_n. We will now see that $\bigcup\limits_{n \in \mathbb{N}} \chi_n(X, \overline{\mathbb{N}})$ is a (countable) *basis* for $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ in the sense of Definition 1.3.8.

Lemma 6.1.17. *Let X be the circle or the interval and let $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. For any $g \in \chi_f(X, \overline{\mathbb{N}})$ there exists $h \in \bigcup\limits_{n \in \mathbb{N}} \chi_{n,f}(X, \overline{\mathbb{N}})$ such that $g \ll h$.*

Proof. Let $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. First, observe that for any $g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ such that $g \ll f$, we can find $f' \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ such that $g \ll f' \ll f \leq \infty$. So without loss of generality, we can suppose that $f \ll \infty$. Equivalently, the canonical decomposition of f is a finite sum. Let us show the statement for $f := 1_U$ for some U open set of X and the result follows using Cuntz axiom (O3) combined with Lemma 6.1.14.
Let $g \in \chi_f$ and write $V := \operatorname{supp} g$. We know that there exists a 1-thin cover $\{\overline{V_k}\}_{k=1}^m$ of X such that g is constant on every V_k. Thus V has a finite number of connected components. As in the proof of Lemma 6.1.15, we first suppose that V is a connected open set and we repeat the process to obtain the result.
Since $g \ll f$, we know that $V \ll U$ and that $g_{|V} \in \{0, 1\}$. In fact, we have $V \subseteq \overline{V} \subseteq U$. If $V = U = X$ then it is clear. Else, we know that the inclusions are strict and hence we can find n big enough and $x_r, x_s \in X \setminus (\overset{3^n}{\underset{k=1}{\cup}} U_k)$ such that $x_{r-1}, x_s \in U \setminus \overline{V}$, where $\{\overline{U_k}\}_1^{3^n}$ the canonical 1-thin cover of X of size $1/3^n$. Now consider $V' := U_r \cup (\bigcup_{k=r+1}^{s-1} \overline{U_k}) \cup U_s$. We have that $V \subseteq \overline{V} \subseteq V' \subseteq \overline{V'} \subseteq U$. Then define $h := 1_{V'}$. By construction, we have $h \in \chi_f$ and $V \ll V' \ll U$, that is, $g \ll h \ll f$, which ends the proof. □

Corollary 6.1.18. *For any $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$, there exists a $\ll$-increasing sequence $(g_l)_l$ in $\bigcup_{n\in\mathbb{N}} \chi_{n,f}(X, \overline{\mathbb{N}})$ such that $\sup_{l\in\mathbb{N}} g_l = f$. Equivalently, $\bigcup_{n\in\mathbb{N}} \chi_n(X, \overline{\mathbb{N}})$ is a countable basis of* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

Proof. Combine Proposition 6.1.7 (ii) with Lemma 6.1.17 to get the result. □

Remark 6.1.19. We will often say that $\bigcup_{n\in\mathbb{N}} \chi_n(X, \overline{\mathbb{N}})$ is *dense* in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$

6.2 Cu-metrics

6.2.1. We continue to assume that X is a compact metric space of covering dimension 1. Let S and T be countably-based Cu-semigroups. Distinct (pseudo)metrics have been used in the past to compare Cu-morphisms. We will in the first place define abstract relations on $\mathrm{Hom}_{\mathrm{Cu}}(S, T)$. Later, now that we have described $\mathrm{Lsc}(X, S)$ more specifically, we will construct a Cu-metric on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X, \overline{\mathbb{N}}), T)$ (inspired by [41, Definition 3.1] or [64, §2.3]) to finally introduce a new discrete Cu-semimetric.

Definition 6.2.2. Let S, T be Cu-semigroups. Let Γ be a finite subset of S. We sometimes write $\Gamma \underset{fin}{\subseteq} S$. Let $\alpha : \Gamma \longrightarrow T$. We say that α is a Cu-*partial morphism* if:
(i) For any $g' \ll g$ in Γ, $\alpha(g') \ll \alpha(g)$.

(ii) For any $g_1 + g_2 \ll h$, $\alpha(g_1) + \alpha(g_2) \ll \alpha(h)$.
(iii) If $g_1, g_2, h_1, h_2 \in \Gamma$ satisfy $g_1 + g_2 = h_1 + h_2$ in S, then $\alpha(g_1) + \alpha(g_2) = \alpha(h_1) + \alpha(h_2)$.

Remark 6.2.3. Any restriction of a Cu-morphism to a finite set of the domain, is a Cu-partial morphism.

Definition 6.2.4. Let Γ_1, Γ_2 be finite subsets of S such that $\Gamma_1 \subseteq \Gamma_2 \subseteq S$. Let $\alpha, \beta : \Gamma_2 \longrightarrow T$ be Cu-partial morphisms.
(i) We say that α is $\approx$-equivalent to β on Γ_1, and we write $\alpha \underset{\Gamma_1}{\approx} \beta$, if for any $g' \ll g$ in Γ_1, $\alpha(g') \ll \beta(g)$ and $\beta(g') \ll \alpha(g)$.
(ii) We say that α is $\simeq$-equivalent to β on Γ_1, and we write $\alpha \underset{\Gamma_1}{\simeq} \beta$, if for any $g' \ll g$ in Γ_1, $\alpha(g') \leq \beta(g)$ and $\beta(g') \leq \alpha(g)$.

Remark 6.2.5. Obviously, if $\alpha \underset{\Gamma_1}{\approx} \beta$ then $\alpha \underset{\Gamma_1}{\simeq} \beta$, and often we will use these notions in the context of $\alpha, \beta : S \longrightarrow T$ being Cu-morphisms and Γ a finite subset of S: see e.g Theorem 6.3.8.

6.2.6. Let us define a metric and, in the specific case of the interval or the circle, a semimetric (that is, a metric that does not necessarily satisfy the triangular inequality) on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X, \overline{\mathbb{N}}), T)$. We mention that the first one has been inspired by a similar construction from [41, Definition 3.1].

Definition 6.2.7. Let X be a compact metric space of covering dimension 1 and let T be Cu-semigroup. For any open set U of X, and any $r > 0$, we define an *r-open neighborhood of U*, that we write $U_r := \underset{x \in U}{\cup} B(x, r)$. Now, for any two Cu-morphisms $\alpha, \beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow T$, we define:

$$(i) \quad d_{\mathrm{Cu}}(\alpha, \beta) := \inf\{r > 0 \mid \forall U \in \mathcal{O}(X), \alpha(1_U) \leq \beta(1_{U_r}) \text{ and } \beta(1_U) \leq \alpha(1_{U_r})\}$$

where U_r is an r-open neighborhood of U. We refer to d_{Cu} as the Cu-*metric*.
If moreover X is the circle or the interval, we define:

$$(ii) \quad dd_{\mathrm{Cu}}(\alpha, \beta) := \inf_{n \in \mathbb{N}}\{1/3^n \mid \forall g' \ll g \in \Gamma_n(X), \alpha(g') \leq \beta(g) \text{ and } \beta(g') \leq \alpha(g)\}.$$

We refer to dd_{Cu} as the *discrete* Cu-*semimetric*.
For both constructions, if the infimum defined does not exist, we set the value to ∞.

Remark 6.2.8. Both constructions of Definition 6.2.7 are symmetric, positive and take value 0 on the diagonal of $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X, \overline{\mathbb{N}}), T) \times \mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X, \overline{\mathbb{N}}), T)$. Moreover d_{Cu} satisfies the triangular inequality, which makes it a pseudometric. We will see below that they both satisfy the identity of indiscernibles (that is, $d(x,y) = 0$ if and only if $x = y$). Thus d_{Cu} is actually metric and dd_{Cu} a semimetric.

Proof. Both constructions are clearly symmetric and if $\alpha = \beta$, then both take value 0. We finally have to check that d_{Cu} satisfies the triangle inequality. Let $\alpha,\beta,\gamma \in \mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X,\overline{\mathbb{N}}),T)$. If $d_{Cu}(\alpha,\beta) = r_1$ and $d_{\mathrm{Cu}}(\beta,\gamma) = r_2$, then for any $U \in \mathcal{O}(X)$ we have $\alpha(1_U) \leq \beta(1_{U_{r_1}}) \leq \gamma(1_{U_{r_1+r_2}})$ and that $\gamma(1_U) \leq \beta(1_{U_{r_2}}) \leq \alpha(1_{U_{r_2+r_1}})$. Thus, $d_{\mathrm{Cu}}(\alpha,\gamma) \leq r_1 + r_2$. □

Lemma 6.2.9. *Let X be the circle or the interval. Let $\alpha,\beta : \mathrm{Lsc}(X,\overline{\mathbb{N}}) \longrightarrow T$ be* Cu-*morphisms and let $n \in \mathbb{N}$. Then:*
(i) $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$ if and only if $\alpha \underset{\Gamma_n}{\simeq} \beta$.
(ii) If $\alpha \underset{\Gamma_m}{\simeq} \beta$, then $\alpha \underset{\Gamma_n}{\approx} \beta$ for any $n < m$.
(iii) If $\alpha \underset{\Gamma_n}{\approx} \beta$, then for any $g' \ll g \in \chi_n$, we have $\alpha(g') \ll \beta(g)$ and $\beta(g') \ll \alpha(g)$.
(iv) If $\alpha \underset{\Gamma_n}{\simeq} \beta$, then for any $g' \ll g \in \chi_n$, we have $\alpha(g') \leq \beta(g)$ and $\beta(g') \leq \alpha(g)$.

Proof. Let $\alpha,\beta : \mathrm{Lsc}(X,\overline{\mathbb{N}}) \longrightarrow T$ be Cu-morphisms and let $n \in \mathbb{N}$.
(i) If $\alpha \underset{\Gamma_n}{\simeq} \beta$, we exactly have for any two $g', g \in \Gamma_n(X)$ such that $g' \ll g$, then $\alpha(g') \leq \beta(g)$ and $\beta(g') \leq \alpha(g)$, hence $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$. Conversely, suppose that $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$. We trivially deduce that $\alpha \underset{\Gamma_m}{\simeq} \beta$ for any $m < n$. Now suppose that $\alpha \underset{\Gamma_n}{\not\simeq} \beta$. Then by definition of the infimum, we conclude that $dd_{\mathrm{Cu}}(\alpha,\beta) = 1/3^{n-1}$. Contradiction.
(ii) Let $n < m$ and take $g'', g \in \Gamma_n$ such that $g'' \ll g$. By Lemma 6.1.15, we can find $g' \in \Gamma_m$ such that $g'' \ll g' \ll g$ in Γ_m. A quick computation gives us that $\alpha(g'') \leq \beta(g') \ll \beta(g)$ and $\beta(g'') \leq \alpha(g') \ll \alpha(g)$. Thus we deduce that $\alpha \underset{\Gamma_m}{\approx} \beta$.
(iii) Let $g', g \in \chi_n$ such that $g' \ll g$. Since Γ_n generates χ_n, see Lemma 6.1.15 (i), we know that there exists two unique $\subseteq$-decreasing sequences $(V'_n)_n$ and $(V_n)_n$ of elements in Γ_n such that $g' = \sum_{n=0}^{\infty} 1_{V'_n}$ and $g = \sum_{n=0}^{\infty} 1_{V_n}$. Moreover, from hypothesis, we also have $1_{V'_n} \ll 1_{V_n}$ for any $n \in \mathbb{N}$. Now, using that $\alpha \underset{\Gamma_n}{\approx} \beta$, for any $n \in \mathbb{N}$ we deduce that $\alpha(1_{V'_n}) \ll \beta(1_{V_n})$ and $\beta(1_{V'_n}) \ll \alpha(1_{V_n})$ the conclusion follows. (iv) is proved similarly. □

Proposition 6.2.10. *Let X be the circle or the interval. Let $\alpha,\beta : \mathrm{Lsc}(X,\overline{\mathbb{N}}) \longrightarrow T$ be* Cu-*morphisms and let $n \in \mathbb{N}$. Then:*
(i) If $d_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$, then $\alpha \underset{\Gamma_n}{\simeq} \beta$.
(ii) If $\alpha \underset{\Gamma_n}{\simeq} \beta$, then $d_{\mathrm{Cu}}(\alpha,\beta) \leq 4/3^n \leq 1/3^{n-2}$.

Proof. Let $\mathcal{U} := \{\overline{U_k}\}_1^{3^l}$ be the canonical 1-thin cover of X of size $1/3^n$.
(i) Let $g, g' \in \Gamma_n$ such that $g' \ll g$. Write $V := \operatorname{supp} g, V' := \operatorname{supp} g'$. Both V, V' has a finite number of (open) connected components. As in the proof of Lemma 6.1.15, we first suppose that V, V' are a connected open sets and we repeat the process to obtain the result. Thus we have $V := U_r \cup (\bigcup_{k=r+1}^{s-1} \overline{U_k}) \cup U_s, V' := U_{r'} \cup (\bigcup_{k=r'+1}^{s'-1} \overline{U_k}) \cup U_{s'}$ for some $r' \leq r \leq s \leq s'$. Since $V \ll V'$, either $V = V' = X$ or else $r' < r \leq s < s'$. In both cases, we observe that $V' \ll V'_{1/3^n} \leq V$. Thus we deduce that $\alpha(g) \leq \beta(g')$ and $\beta(g) \leq \alpha(g')$.
(ii) Let $U \in \mathcal{O}(X)$ and consider $f := 1_U$. Let us construct recursively the following $g \in \Gamma_n$:
(1) For any $1 \leq k \leq 3^n$, if $U_k \cup U \neq \emptyset$, put $g_{|U_k} = 1$.
(2) For any $x \in X \setminus (\bigcup_{k=1}^{3^n} U_k)$, put $g(x) = f(x)$.
Write $V := \operatorname{supp} g$. From construction, g is an element of Γ_n such that $U \subseteq V \subseteq U_{1/3^n} \subseteq V_{1/3^n} \subseteq U_{2/3^n}$. Since $1_V, 1_{V_{2/3^n}}$ are elements of Γ_n such that $1_V \ll 1_{V_{1/3^n}}$, we deduce the following by using that $\alpha \underset{\Gamma_n}{\simeq} \beta$:

$$\begin{cases} \alpha(1_{|U}) \leq \alpha(1_{|V}) \leq \beta(1_{V_{1/3^n}}) \leq \beta(1_{U_{2/3^n}}) \\ \beta(1_{|U}) \leq \beta(1_{|V}) \leq \alpha(1_{V_{1/3^n}}) \leq \alpha(1_{U_{2/3^n}}) \end{cases}$$

which ends the proof. □

6.2.11. *Conjecture:* Let X be the circle or the interval. Let $\alpha,\beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow T$ be Cu-morphisms. Then, $dd_{\mathrm{Cu}}(\alpha,\beta) \leq d_{\mathrm{Cu}}(\alpha,\beta) \leq 3dd_{\mathrm{Cu}}(\alpha,\beta)$.

Corollary 6.2.12. *Let X be a compact metric space of covering dimension 1. Let α,β : $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow T$ be* Cu-*morphisms. The following are equivalent:*
(i) $d_{\mathrm{Cu}}(\alpha,\beta) = 0$.
(ii) $dd_{\mathrm{Cu}}(\alpha,\beta) = 0$. (Here we suppose X to be the circle or the interval.)
(iii) $\alpha \underset{\Gamma_n}{\simeq} \beta$, for any $n \in \mathbb{N}$.
(iii') $\alpha \underset{\Gamma_n}{\approx} \beta$, for any $n \in \mathbb{N}$.
(iv) $\alpha = \beta$.
Hence d_{Cu} is a metric and dd_{Cu} is a semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Lsc}(X, \overline{\mathbb{N}}), T)$.

Proof. Let $\alpha,\beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow T$ be Cu-morphisms. That (iv) implies (i) trivially. Conversely, suppose that $d_{\mathrm{Cu}}(\alpha,\beta) = 0$. Let $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$. Since $B := \{1_U\}_{U \in \mathcal{O}(X)}$ generates $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ (see Remark 6.1.16), we can suppose that $f := 1_U$ for some $U \in \mathcal{O}(X)$. Now using (O2), we know there exists a $\ll$- increasing sequence $(g_n)_n$ such that $\sup_{n \in \mathbb{N}} g_n = f$. A fortiori,

$g_n \in B$ for any $n \in \mathbb{N}$. Now from hypothesis, we know that $\alpha(g_m) \leq \beta(g_n) \leq \beta(f)$ and $\beta(g_m) \leq \alpha(g_n) \leq \alpha(f)$ for any $m < n$. Finally, passing to suprema, we obtain that $\alpha(f) \leq \beta(f)$ and $\beta(f) \leq \alpha(f)$, which gives us that (i) is equivalent to (iv).
By Lemma 6.2.9, we directly obtain (ii) is equivalent to (iii) is equivalent to (iii') and Proposition 6.2.10 exactly gives us (i) is equivalent to (ii), which ends the proof. □

6.3 Intertwinings

6.3.1. In this third section, we exhibit a theorem that is a particular case of approximate intertwinings in the category Cu in a specific countably-based setting. It is worth mentioning that it has been inspired from the work of Thomsen in [73, Theorem 3.4]. Let us first start with a characterization regarding inductive systems in the category Cu and some more properties of the n-piecewise characteristic functions:

Proposition 6.3.2. *[65, Section 3],[21, Theorem 2]*
Consider an inductive system $(S_i, \sigma_{ij})_{i\in I}$ in Cu. *Then $(S, \sigma_{i\infty})_{i\in I}$ is the inductive limit of the system if and only if it satisfies the two following properties:*
(L1): For any $s \in S$, there exists $(s_i)_{i\in I}$ such that $s_i \in S_i$ for any i and $\sigma_{i(i+1)}(s_i) \ll s_{i+1}$ for any $i \in I$ and $s = \sup\limits_{i\in I} \sigma_{i\infty}(s_i)$.
(L2): Let s, t be elements in S_i such that $\sigma_{i\infty}(s) \leq \sigma_{i\infty}(t)$, and $s' \ll s$. Then there exists $j \geq i$ such that $\sigma_{ij}(s') \ll \sigma_{ij}(t)$.

Lemma 6.3.3. *Let X be the circle or the interval and consider the* Cu*-semigroup* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ *and $l \in \mathbb{N}$. Let $U \in \mathcal{O}(X)$ and consider $f := 1_{|U}$. Then $\Gamma_{l,f}$ has a largest element, that we write $f_{\ll,l}$. That is, $f_{\ll,l} := \max\limits_{g\in\Gamma_{l,f}}\{g \ll f\}$.*

Proof. We explicitly build the largest element of the set $\Gamma_{l,f}$. Write $\{\overline{U_k}\}_{k=1}^{3^l}$ the canonical 1-thin cover of X of size $1/3^l$. Recursively, construct $f_{\ll,l}$ as follows:
(1) For any $0 \leq k \leq 3^l$, if $\overline{U_k} \cap U = \overline{U_k}$, then put $f_{\ll,l|U_k} = 1$. Else, put $f_{\ll,l|U_k} = 0$.
(2) For any $x \in X \setminus (\bigcup\limits_{k=1}^{3^l} U_k)$, if there exists an open neighborhood B_x of x such that $f_{\ll,l|(B_x\setminus\{x\})} = 1$, then put $f_{\ll,l}(x) = 1$, else put $f_{\ll,l}(x) = 0$. By construction, $f_{\ll,l} \in \Gamma_{l,f}$ and has the required property. □

Corollary 6.3.4. *Let X be the circle or the interval. Let $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ such that $f \ll \infty$ and let $(W_k)_k$ its canonical decomposition. Let $l \in \mathbb{N}$. Then $\chi_{l,f}$ has a largest element, that we write*

$f_{\ll,l}$. *In fact,* $f_{\ll,l} := \max\{g \ll f, g \in \chi_l\} = \sum_k (1_{W_k})_{\ll,l}$.
Furthermore, $(f_{\ll,l})_{l\in\mathbb{N}}$ *is a* $\ll$*-increasing sequence whose supremum is* f.

Proof. Let $g \in \chi_{f,l}$. Combining Lemma 6.1.14 (iii), with Lemma 6.1.15 (i) we get that the canonical decomposition $(V_k)_k$ of g satisfies the following: $V_k \in \Gamma_{l,f}$ and $V_k \ll W_k$ for any k. On the other hand, using Lemma 6.3.3 and Lemma 6.1.14 (ii), we deduce that $g \leq \sum_k (1_{W_k})_{\ll,l}$. Now, by Lemma 6.1.15, we know that we can find some $h \in \chi_{l+1}$ such that $f_{\ll,l} \ll h \ll f$. We obtain $(f_{\ll,l})_{l\in\mathbb{N}}$ is a $\ll$-increasing sequence. Thus, by Corollary 6.1.18, we can find a $\ll$-increasing sequence $(g_n)_n$ in $\bigcup_{l\in\mathbb{N}} \chi_l$ whose supremum is f. Since any $g_n \in \chi_l$ for some $l \in \mathbb{N}$, we have $g_n \leq f_{\ll,l} \leq \sup_l f_{\ll,l}$ for any $n \in \mathbb{N}$. Passing to suprema, we deduce $f \leq \sup_l f_{\ll,l}$ and obviously the converse inequality holds. □

Lemma 6.3.5. *Let X be the circle or the interval. Let $l \in \mathbb{N}$. Let $g, g' \in \Gamma_l$, and put $h := g + g' \in \chi_l$. Then for any $j > l$, we have $g_{\ll,j} + g'_{\ll,j} = h_{\ll,j}$.*
Now write $V := \operatorname{supp} g$, $V' := \operatorname{supp} g'$ and write $V_{\ll,j} := \operatorname{supp} g_{\ll,j}$, $(V')_{\ll,j} := \operatorname{supp} g'_{\ll,j}$. Then $(V \cup V')_{\ll,j} = (V_{\ll,j} \cup (V')_{\ll,j})$ and $(V \cap V')_{\ll,j} = (V_{\ll,j} \cap (V')_{\ll,j})$.

Proof. Let $\mathcal{U} := \{\overline{U_k}\}_1^{3^l}$ be the canonical 1-thin cover of X of size $1/3^l$. Let $h := g + g'$, $V := \operatorname{supp} g$ and $V' := \operatorname{supp} g'$. Also write $h_1 := 1_{V\cup V'}$ and $h_2 := 1_{V\cap V'}$. By Lemma 6.3.3, we know that $h = h_1 + h_2$. Further, both V, V' have a finite number of (open) connected components. As in the proof of Lemma 6.1.15, we first suppose that V, V' are connected open sets and we repeat the process to obtain the result. If $V \cap V' = \emptyset$ then $h \in \Gamma_l$ and the result is trivial. Else $V \cap V' \neq \emptyset$. In this case we have:

$$V := U_r \cup (\bigcup_{k=r+1}^{s-1} \overline{U_k}) \cup U_s \qquad\qquad V' := U_{r'} \cup (\bigcup_{k=r'+1}^{s'-1} \overline{U_k}) \cup U_{s'}$$

for some $r \leq r' \leq s \leq s'$.
Let $j > l$ and and let $\mathcal{W} := \{\overline{W_i}\}_1^{3^j}$ be the canonical 1-thin cover of X of size $1/3^j$. Observe that for any $1 \leq k \leq 3^l$, we can find 3^{j-l} open sets $W_{i,k}$ of $\mathcal{W}$ such that $W_{i,k} \subseteq U_k$ and such that $\{\overline{W_{i,k}}\}_{i=1}^{3^{j-l}}$ is a 1-thin cover of $\overline{U_k}$ of size $1/3^{j-l}$. Now observe that:

$$\operatorname{supp}(h_1)_{\ll,j} = (U_r \setminus \overline{W_{r,1}}) \cup (\bigcup_{k=r+1}^{s'-1} \overline{U_k}) \cup (U_{s'} \setminus W_{s',3^{j-l}})$$
$$\operatorname{supp}(h_2)_{\ll,j} = (U_{r'} \setminus \overline{W_{r',1}}) \cup (\bigcup_{k=r'+1}^{s-1} \overline{U_k}) \cup (U_s \setminus W_{s,3^{j-l}})$$

Moreover, since $(V \cup V', V \cap V')$ is the canonical decomposition of h, by Corollary 6.3.4, we know that $h_{\ll,n_j} = (h_1)_{\ll,j} + (h_2)_{\ll,j}$. On the other, observe that:

$$\operatorname{supp}(g)_{\ll,j} = (U_r \setminus \overline{W_{r,1}}) \cup (\bigcup_{k=r+1}^{s-1} \overline{U_k}) \cup (U_s \setminus W_{s,3^{j-l}})$$
$$\operatorname{supp}(g')_{\ll,j} = (U_{r'} \setminus \overline{W_{r',1}}) \cup (\bigcup_{k=r'+1}^{s'-1} \overline{U_k}) \cup (U_{s'} \setminus W_{s',3^{j-l}})$$

Hence we get $h_{\ll,j} = g_{\ll,j} + g'_{\ll,j}$.
Now, let us write $V_{\ll,j} := \operatorname{supp} g_{\ll,j}, (V')_{\ll,j} := \operatorname{supp} g'_{\ll,j}$. Observe that by Corollary 6.3.4, we know that $g_{\ll,j} + (g')_{\ll,j} = 1_{V_{\ll,j} \cup (V')_{\ll,j}} + 1_{V_{\ll,j} \cap (V')_{\ll,j}}$. On the other hand, we have just proved that $g_{\ll,j} + (g')_{\ll,j} = (h_1)_{\ll,j} + (h_2)_{\ll,j} = 1_{(V \cup V')_{\ll,j}} + 1_{(V \cap V')_{\ll,j}}$, which ends the proof. □

Corollary 6.3.6. *Let X be the circle or the interval and consider the* Cu-*semigroup* $\operatorname{Lsc}(X, \overline{\mathbb{N}})$. *Let $l \in \mathbb{N}$ and let $g, g' \in \chi_l$.*
(i) For any $j > l$, we have $g_{\ll,j} + g'_{\ll,j} = (g + g')_{\ll,j}$.
(ii) Suppose $g' \ll g$, (respectively $g' \leq g$). Then for any $j > l$, we have $g'_{\ll,j} \ll g_{\ll,j}$, (respectively $g'_{\ll,j} \leq g_{\ll,j}$ for any $j \in \mathbb{N}$).
(iii) Let $(g_k)_k, (h_k)_k$ be $\ll$-increasing sequences in $\bigcup_{l \in \mathbb{N}} \chi_l$ such that $g := \sup_{k \in \mathbb{N}}(g_k) = \sup_{k \in \mathbb{N}}(h_k)$. Let $j \in \mathbb{N}$. Then, $((g_k)_{\ll,j})_{k \in \mathbb{N}}, ((h_k)_{\ll,j})_{k \in \mathbb{N}}$ are increasing sequences in S_i and we have $\sup_{k \in \mathbb{N}}(g_k)_{\ll,j} = \sup_{k \in \mathbb{N}}(h_k)_{\ll,j}$. In particular, if $g \ll \infty$, then $\sup_{k \in \mathbb{N}}(g_k)_{\ll,j} = g_{\ll,j}$.

Proof. Since $g' \in \chi_l$ is a sum of elements in Γ_l, we can suppose without loss of generality that $g' \in \Gamma_l$. Now let $(V_n)_n$ be the canonical decomposition of g and let $V' := \operatorname{supp} g'$. we have $g = \sum_n 1_{V_n}$ and $g' := 1_{V'}$. Observe that $g + g' = 1_{V_1 \cup V'} + \sum_n 1_{V_n \cup (V_{n-1} \cap V')}$ (respectively, $g_{\ll,j}, g'_{\ll,j}$ and their canonical decompositions). Now that we know the canonical decomposition of $g+g'$ and $g_{\ll,j} + g'_{\ll,j}$, using Corollary 6.3.4 and Lemma 6.3.5 (and its notations), we get that:

$$\begin{aligned} g_{\ll,j} + g'_{\ll,j} &= \sum_n (1_{V_n})_{\ll,j} + (1_{V'})_{\ll,j} \\ &= 1_{(V_1)_{\ll,j} \cup (V')_{\ll,j}} + \sum_{n \geq 2} 1_{(V_n)_{\ll,j} \cup ((V_{n-1})_{\ll,j} \cap (V')_{\ll,j})} \\ &= 1_{(V_1 \cup V')_{\ll,j}} + \sum_{n \geq 2} 1_{(V_n)_{\ll,j} \cup (V_{n-1} \cap V')_{\ll,j}} \end{aligned}$$

$$= (1_{(V_1 \cup V')})_{\ll,j} + \sum_{n\geq 2} 1_{(V_n \cup (V_{n-1} \cap V'))_{\ll,j}}$$
$$= (1_{(V_1 \cup V')})_{\ll,j} + \sum_{n\geq 2} 1_{(V_n \cup (V_{n-1} \cap V'))_{\ll,j}}$$

and hence $g_{\ll,j} + g'_{\ll,j} = (g + g')_{\ll,j}$.
(ii) Now assume that $g' \ll g$. Let $j > l$ Using Lemma 6.1.15, we know that we can find some $h \in \chi_j$ such that $g' \ll h \ll g$. From this, we deduce that $(g')_{\ll,j} \ll g_{\ll,j}$. If we have $g' \leq g$, then since $g'_{\ll,j} \ll g' \leq g$, the result follows from maximality of $g'_{\ll,j}$ for any $j \in \mathbb{N}$.
(iii) Let $(g_k)_k, (h_k)_k$ be $\ll$-increasing sequences in $\bigcup_{l\in\mathbb{N}} \chi_l$ such that $g := \sup_{k\in\mathbb{N}}(g_k) = \sup_{k\in\mathbb{N}}(h_k)$. Let $j \in \mathbb{N}$. We know that $(g_k)_{\ll,j} \ll g_k \ll g_{k+1}$. Hence, $(g_k)_{\ll,j} \in \chi_{j,g_{k+1}}$. By Corollary 6.3.4 we know that $\chi_{j,g_{k+1}}$ has a largest element, which by definition is $(g_{k+1})_{\ll,j}$. We finally deduce that $(g_k)_{\ll,j} \leq (g_{k+1})_{\ll,j}$. A similar argument gives us $(h_k)_{\ll,j} \leq (h_{k+1})_{\ll,j}$ and we deduce that both $((g_k)_{\ll,j})_k, ((h_k)_{\ll,j})_k$ are increasing sequences in χ_j. Thus, they have suprema that we respectively denote by $\tilde{g}_j, \tilde{h}_j$.
Since $(g_k)_k, (h_k)_k$ are $\ll$-increasing and $\sup_{k\in\mathbb{N}}(g_k) = \sup_{k\in\mathbb{N}}(h_k)$, we know that for any $k \in \mathbb{N}$, there exists $k' \in \mathbb{N}$ such that $g_k \ll h_{k'}$ and $h_k \ll g_{k'}$. We deduce that for any $k \in \mathbb{N}$, there exists a $k' \in \mathbb{N}$ such that $(g_k)_{\ll,j} \leq (h_{k'})_{\ll,j}$ and $(h_k)_{\ll,j} \leq (g_{k'})_{\ll,j}$. Passing to suprema, we conclude that $\tilde{g}_j = \tilde{h}_j$.
Finally, if $g \ll \infty$, then $((g_k)_{\ll,j})_{k\in\mathbb{N}}$ is in fact an increasing sequence in $\chi_{j,g}$, which happens to have $g_{\ll,j}$ as largest element. Thus we get that $\tilde{g}_j \leq g_{\ll,j}$. Conversely, since $g_{\ll,j} \ll g$, we know that there exists some $k \in \mathbb{N}$ such that $g_{\ll,j} \leq g_k \ll g_{k+1}$, which gives us that $g_{\ll,j} \leq (g_{k+1})_{\ll,j}$. We deduce that $g_{\ll,j} \leq \tilde{g}_j$, hence $g_{\ll,j} = \tilde{g}_j$ and the result follows. □

Lemma 6.3.7. *Let S, T be* Cu*-semigroups. Suppose that S has a countable basis B such that B is also a* PoM*. For any* PoM*-morphism $\alpha : B \longrightarrow T$, then there exists a generalized* Cu*-morphism $\tilde{\alpha} : S \longrightarrow T$ (that is, a PoM-morphism that respects suprema of increasing sequences), such that $\tilde{\alpha}_{|B} = \alpha$. If moreover α preserves $\ll$, then $\tilde{\alpha}$ is a* Cu*-morphism.*
For notational purposes, we usually write α instead of $\tilde{\alpha}$.

Proof. Let $\alpha : B \longrightarrow T$ be a PoM-morphism. We are going to extend α to S: Let $(b_n)_n, (c_n)_n$ be two $\ll$-increasing sequences in B such that they have the same supremum in S. That is, $\sup_n b_n = \sup_n c_n$ in S. First, observe that $\alpha(b_n)_n, \alpha(c_n)_n$ are increasing sequences in T and hence they have a supremum. Also, since $(b_n)_n$ is a $\ll$-increasing sequence, it follows that for any $n \in \mathbb{N}$, there exists some $m \geq n$ such that $b_n \leq c_m$. Besides α is a PoM-morphism, so $\alpha(b_n) \leq \alpha(c_m) \leq \sup_n \alpha(c_n)$, for all $n \in \mathbb{N}$. It follows that $\sup_n \alpha(b_n) \leq \sup_n \alpha(c_n)$. By symmetry,

we get the converse inequality to conclude: $\sup_n \alpha(b_n) = \sup_n \alpha(c_n)$ for any two $\ll$-increasing sequences of B that have the same supremum in S.
Now let $s \in S$. Since B is a (countable) basis of S, then there exists a $\ll$-increasing sequence $(b_n)_n$ in B whose supremum is s. We have just proved that $\sup_n \alpha(b_n)$ does not depend on the sequence $(b_n)_n$ chosen. Thus we define:

$$\begin{aligned} \alpha : S &\longrightarrow T \\ s &\longmapsto \sup_n \alpha(b_n) \end{aligned}$$

Let us check that this is a generalized Cu-morphism. Let $s, t \in S$, and consider two $\ll$-increasing sequences $(b_n)_n, (c_n)_n$ in B such that $s = \sup_n b_n, t = \sup_n c_n$.
Suppose that $s \leq t$. We have $\sup_n b_n \leq \sup_n c_n$ and we have proved just above that this implies that $\sup_n \alpha(b_n) \leq \sup_n \alpha(c_n)$. That is, $\alpha(s) \leq \alpha(t)$.
Now put $x := s + t \in S$. Then $(b_n + c_n)_n$ is a sequence of B, since B is a PoM, and obvisouly, it is $\ll$-increasing towards x. Hence $\alpha(x) = \sup_n \alpha(b_n + c_n) = \sup_n(\alpha(b_n) + \alpha(c_n))$. Further, we know that $\sup_n(\alpha(b_n), \sup_n(\alpha(c_n)$ exist in T. In fact, they are respectively equal to $\alpha(s), \alpha(t)$. Thus we deduce that $\alpha(x) = \alpha(s)+\alpha(t)$ and that $\alpha : S \longrightarrow T$ is a well-defined PoM-morphism.
Let $(s_n)_n$ be an increasing sequence in S. Then $(\alpha(s_n))_n$ is an increasing sequence in T and hence it has a supremum. On the one hand, we have $\alpha(s_n) \leq \alpha(s)$ for any $n \in \mathbb{N}$, from which we obtain $\sup_n(\alpha(s_n)) \leq \alpha(s)$. On the other hand, we know we can find a $\ll$-increasing sequence $(b_n)_n$ in B such that $s = \sup_n b_n$ and that for any b_n, there exists m such that $b_n \leq s_m$, which gives us $\alpha(b_n) \leq \alpha(s_m) \leq \sup_n(\alpha(s_n))$ for any $n \in \mathbb{N}$. Passing to suprema, we deduce the converse inequality: $\alpha(s) = \sup_n(\alpha(b_n)) \leq \sup_n(\alpha(s_n))$. We conclude that $\alpha : S \longrightarrow T$ is a well-defined generalized Cu-morphism.
Finally, suppose that $\alpha : B \longrightarrow T$ preserves $\ll$. Then for any two $s, t \in S$ such that $s \ll t$, consider any $\ll$-increasing sequence $(c_n)_n$ in B whose supremum is t. Then we can find some $m \in \mathbb{N}$ such that $s \ll c_m \ll c_{m+1} \ll t$. We obtain: $\alpha(s) \leq \alpha(c_m) \ll \alpha(c_{m+1}) \leq \alpha(t)$, which gives us that $\alpha : S \longrightarrow T$ is a Cu-morphism. □

Theorem 6.3.8. *Let X be the circle or the interval. For any $i \in \mathbb{N}$, write $S_i = T_i =$ Lsc$(X, \overline{\mathbb{N}})$. Suppose there are* Cu*-morphisms $\sigma_{ii+1}, \tau_{ii+1}$: Lsc$(X, \overline{\mathbb{N}}) \longrightarrow$ Lsc$(X, \overline{\mathbb{N}})$ that define two inductive sequences $(S_i, \sigma_{ij})_{i\in\mathbb{N}}$ and $(T_i, \tau_{ij})_{i\in\mathbb{N}}$ in* Cu. *Let S and T be their respective inductive limits. Suppose there exists a strictly increasing sequence $(n_i)_i$ in $\mathbb{N}$ such that:*

(i) $\sigma_{ii+1} \underset{\Gamma_{n_i}}{\approx} \tau_{ii+1}$, *for any* $i \in \mathbb{N}$.
(ii) For any $i \leq j$ *and any* $l \in \mathbb{N}$, *we have* $\sigma_{ij}(\chi_l), \tau_{ij}(\chi_l) \subseteq \chi_l$, *where* $\sigma_{ij} := \sigma_{j-1j} \circ ... \circ \sigma_{ii+1}$ *and* $\tau_{ij} := \tau_{j-1j} \circ ... \circ \tau_{ii+1}$.
Then, $S \simeq T$ *as* Cu-*semigroups.*

Proof. The aim is to construct generalized Cu-morphisms $\gamma_i : S_i \longrightarrow T$, for any $i \in \mathbb{N}$, such that $\gamma_i = \gamma_j \circ \sigma_{ij}$ for any $i \leq j$, to obtain a generalized Cu-morphism $\gamma : S \longrightarrow T$. A similar construction will give us another generalized Cu-morphism $\delta : T \longrightarrow S$. Finally, we will conclude by showing that γ and δ are in fact Cu-isomorphisms inverses of one another.

Fix $i \in \mathbb{N}$. We will first define morphisms $\gamma_{i,n_l} : \chi_{n_l} \subset S_i \longrightarrow T$, for any $l \in \mathbb{N}$ as follows:
Let $l \in \mathbb{N}$ and let $g \in \chi_{n_l}$. For any $j > i, n_l$ we define:

$$\begin{array}{ccccc} \alpha_j : & S_i & \xrightarrow{\sigma_{ij+1}} & S_{j+1} & \\ & & & \downarrow{\scriptstyle id} & \\ & & & T_{j+1} & \xrightarrow[\tau_{j+1\infty}]{} T \end{array}$$

We claim that $(\alpha_j(g_{\ll,n_j}))_{j>i,n_l}$ is a $\ll$ increasing sequence in T. Indeed, by Corollary 6.3.4 and because $(n_i)_i$ is a strictly increasing sequence, we know that $(g_{\ll,n_j})_{j\in\mathbb{N}}$ is a $\ll$-increasing sequence in S_i whose supremum is g. We deduce that $\sigma_{ij+1}(g_{\ll,n_j}) \ll \sigma_{ij+1}(g_{\ll,n_{j+1}})$ in S_{j+1} for any $j > i, n_l$.
Moreover, by (ii), we know that $\sigma_{ij+1}(g_{\ll,n_j}), \sigma_{ij+1}(g_{\ll,n_{j+1}}) \in \chi_{n_{j+1}}$. Now using Lemma 6.2.9 (iii), combined with (i), we get that $\tau_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_j}) \ll \sigma_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_{j+1}})$ and hence $\tau_{j+2\infty} \circ \tau_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_j}) \ll \tau_{j+2\infty} \circ \sigma_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_{j+1}})$. That is, $\alpha_j(g_{\ll,n_j}) \ll \alpha_{j+1}(g_{\ll,n_{j+1}})$, for any $j > i, n_l$. Thus, $(\alpha_j(g_{\ll,n_j}))_{j>i,n_l}$ is a $\ll$ increasing sequence in T and it has a supremum.
Let us now define:

$$\begin{aligned} \gamma_{i,n_l} : \chi_{n_l} \subseteq S_i &\longrightarrow T \\ g &\longmapsto \sup_{j>i,n_l} \alpha_j(g_{\ll,n_j}) \end{aligned}$$

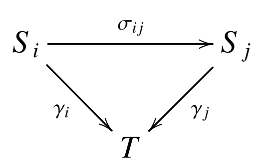
Let us check that this is a well-defined PoM-morphism:
(1) From Corollary 6.3.6, we deduce that for any $g, g', h, h' \in \chi_{n_l}$ such that $g + g' = h + h' := f$, we have $g_{\ll,n_j} + g'_{\ll,n_j} = h_{\ll,n_j} + (h')_{\ll,n_j} = f_{\ll,n_j}$. So we can naturally consider $\gamma_{i,n_l}(g + g') := \gamma_{i,n_l}(g) + \gamma_{i,n_l}(g')$ for any two $g, g' \in \chi_{n_l}$.
(2) Let $g', g \in \chi_{n_l}$ be such that $g' \leq g$. Let $j > i, n_l$. From Corollary 6.3.6 (ii), we know that

$g'_{\ll,n_j} \leq g_{\ll,n_j}$. Hence $\alpha_j(g'_{\ll,n_j}) \leq \alpha_j(g_{\ll,n_j})$ and we conclude that $\gamma_{i,n_l}(g') \leq \gamma_{i,n_l}(g)$.

For any $l \in \mathbb{N}$, we have constructed a PoM-morphism $\gamma_{i,n_l} : \chi_{n_l} \subseteq S_i \longrightarrow T$. Clearly, we have $(\gamma_{i,n'_l})_{|\Gamma_{n_l}} = \gamma_{i,n_l}$ for any $l < l'$. Thus we define the following PoM-morphism:

$$\begin{aligned} \gamma_i : \bigcup_{l\in\mathbb{N}} \chi_{n_l} \subset S_i &\longrightarrow T \\ g &\longmapsto \gamma_{i,n_g}(g) \end{aligned}$$

where $n_g := \min_{l\in\mathbb{N}}\{n_l : g \in \chi_{n_l}\}$.
Finally, by (iii), observe that $\bigcup_{l\in\mathbb{N}} \chi_l = \bigcup_{l\in\mathbb{N}} \chi_{n_l}$. Thus, from Corollary 6.1.18, we deduce that $\bigcup_{l\in\mathbb{N}} \chi_{n_l}$ is a (countable) basis of S_i. Now, combining this with the fact that $\bigcup_{l\in\mathbb{N}} \chi_{n_l}$ is a PoM (see Section 6.1) and that γ_i is a PoM-morphism, we can use Lemma 6.3.7 to extend this map to S_i to obtain a generalized Cu-morphism $\gamma_i : S_i \longrightarrow T$. We recall that, for $f \in S_i$, we have $\gamma_i(f) := \sup_n \gamma_i(g_n)$, where $(g_n)_n$ is any $\ll$-increasing sequence of $\bigcup_{l\in\mathbb{N}} \chi_{n_l}$ whose supremum is f.
To conclude the first step, we have defined generalized Cu-morphisms $\gamma_i : S_i \longrightarrow T$ for any $i \in \mathbb{N}$. Furthermore, using hypothesis (ii), the following diagram is clearly commutative by construction:



Let us now use these γ_i to build a generalized Cu-morphism $\gamma : S \longrightarrow T$. We define:

$$\begin{aligned} \gamma : S &\longrightarrow T \\ s &\longmapsto \sup_i(\gamma_i(s_i)) \end{aligned}$$

where $(s_i)_{i\in\mathbb{N}}$ is any sequence as in (L1).
Let $s \in S$. Consider two sequences $(s_i)_{i\in\mathbb{N}}, (s'_i)_{i\in\mathbb{N}}$ as in (L1) of Proposition 6.3.2. First observe that for any $i \in \mathbb{N}$, we have $\gamma_i(s_i) = \gamma_{i+1}(\sigma_{ii+1}(s_i)) \leq \gamma_{i+1}(s_{i+1})$, respectively $\gamma_i(s'_i) \leq \gamma_{i+1}(s'_{i+1})$. It follows that $(\gamma_i(s_i))_i, (\gamma_i(s'_i))_i$ are increasing sequences in T.
On the other hand, we know that $(\sigma_{i\infty}(s_i))_i, (\sigma_{i\infty}(s'_i))_i$ are $\ll$-increasing sequences whose supremum is s. Let $i \in \mathbb{N}$. It is easy to see that there exists $j \in \mathbb{N}$ such that $\sigma_{i+1\infty}(s_{i+1}) \ll \sigma_{j\infty}(s'_j)$. By (L2) of Proposition 6.3.2, there exists a finite stage $k > i, j$ such that $\sigma_{jk}(\sigma_{ij}(s_i)) \ll \sigma_{jk}(s'_j)$. Since $\gamma_k : S_k \longrightarrow T$ is a generalized Cu-morphism, we obtain $\gamma_k(\sigma_{jk}(\sigma_{ij}(s_i))) \leq \gamma_k(\sigma_{jk}(s'_j))$, which gives us by the commutative diagram above, $\gamma_i(s_i) \leq \gamma_j(s'_j)$.

Thus, we obtain that $\gamma_i(s_i) \leq \gamma_j(s'_j) \leq \sup\limits_l(\gamma_l(s'_l))$ for any $i \in \mathbb{N}$. Passing to suprema, we deduce $\sup\limits_l(\gamma_l(s_l)) \leq \sup\limits_l(\gamma_l(s'_l))$. By a similar argument, the converse inequality holds and we finally conclude:
For any $s \in S$ and any sequence $(s_i)_{i\in\mathbb{N}}$ as in (L1) of Proposition 6.3.2, then $\sup\limits_i(\gamma_i(s_i))$ does not depend on the sequence chosen. By a similar argument as in Lemma 6.3.7, $\gamma : S \longrightarrow T$ is a generalized Cu-morphism such that the following diagram is commutative:

$$\begin{array}{ccc} S_i & \xrightarrow{\sigma_{i\infty}} & S \\ & {\scriptstyle\gamma_i}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & T & \end{array}$$

for any $i \in \mathbb{N}$. We construct in the exact same way, a collection of generalized Cu-morphisms $\{\delta_i\}_{i\in\mathbb{N}}$ and $\delta : T \longrightarrow S$ such that the following diagram is commutative:

$$\begin{array}{ccc} T_i & \xrightarrow{\tau_{i\infty}} & T \\ & {\scriptstyle\delta_i}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & S & \end{array}$$

The last step is then to check that γ and δ are inverses of one another. We are going to prove that for any $i \in \mathbb{N}$ and any $s \in S_i$, we have $\delta \circ \gamma_i(s) = \sigma_{i\infty}(s)$. Since $\bigcup\limits_{l\in\mathbb{N}} \chi_{n_l}$ is dense in S_i and since $\sigma_{i\infty}, \delta, \gamma_i$ are morphisms that preserve suprema of increasing sequences, it is enough to show the said property for some $g \in \bigcup\limits_{l\in\mathbb{N}} \chi_{n_l}$.

Fix $i, l \in \mathbb{N}$ and let $g \in \chi_{n_l}$. The aim is to prove that $\delta\circ\gamma_i(g) \leq \sigma_{i\infty}(g)$ and that $\delta\circ\gamma_i(g) \geq \sigma_{i\infty}(g)$, which will end the proof as explained above. To begin, we observe the following:

$$\begin{aligned} \delta \circ \gamma_i(g) &= \delta(\sup_{j>i,n_g} \tau_{j+1\infty} \circ \sigma_{ij+1}(g_{\ll,n_j})) \\ &= \sup_{j>i,n_g} (\delta \circ \tau_{j+1\infty} \circ \sigma_{ij+1}(g_{\ll,n_j})) \\ &= \sup_{j>i,n_g} \delta_{j+1}(\sigma_{ij+1}(g_{\ll,n_j})) \\ \delta \circ \gamma_i(g) &= \sup_{j>i,n_g} \sup_{j'>j+1,n_j} (\sigma_{j'+1\infty} \circ \tau_{j+1j'+1})((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}). \end{aligned}$$

We first prove that $\delta \circ \gamma_i(g) \leq \sigma_{i\infty}(g)$:

By Corollary 6.3.4, we know that $(g_{\ll,n_j})_{j>i,n_g}$ is a $\ll$-increasing sequence whose supremum is g. By the same argument, we have that $((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}})_{j'>j+1,n_j}$ is a $\ll$-increasing sequence whose supremum is $\sigma_{ij+1}(g_{\ll,n_j})$. Fix $j > i, n_g$, there exists $j' > j+1, n_j$ such that:

$$(\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}} \ll \sigma_{ij+1}(g_{\ll,n_j}) \ll \sigma_{ij+1}(g_{\ll,n_{j+1}}) \text{ in } S_{j+1}.$$

Furthermore, by hypothesis (ii), observe that $\sigma_{ij+1}(g_{\ll,n_j}) \ll \sigma_{ij+1}(g_{\ll,n_{j+1}})$ are in $\chi_{n_{j+1}} \subseteq S_{j+1}$. Using that τ_{j+1j+2} is a Cu-morphism on the first comparison and hypothesis (i) on the second comparison, we deduce the following:

$$\tau_{j+1j+2}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \ll \tau_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_j}) \ll \sigma_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_{j+1}}) \text{ in } S_{j+2}.$$

Again, by hypothesis (ii), observe that $\tau_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_j}), \sigma_{j+1j+2} \circ \sigma_{ij+1}(g_{\ll,n_{j+1}}) \in \chi_{n_{j+2}} \subseteq S_{n_{j+2}} = T_{n_{j+2}}$. Hence, hypothesis (i) still applies here, and repeating this process we obtain for any $j' > j+1, n_j$:

$$\tau_{j+1j'+1}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \ll \tau_{j+1j'+1}(\sigma_{ij+1}(g_{\ll,n_j})) \ll \sigma_{j+1j'+1}(\sigma_{ij+1}(g_{\ll,n_{j+1}})) \text{ in } S_{j'+1}.$$

Composing with $\sigma_{j'+1\infty}$, we obtain for any $j' > j+1, n_j$:

$$\sigma_{j'+1\infty} \circ \tau_{j+1j'+1}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \ll \sigma_{j'+1\infty} \circ \sigma_{j+1j'+1}(\sigma_{ij+1}(g_{\ll,n_{j+1}})) \text{ in } S.$$

Now, taking suprema over j' first and then over j, we conclude that $\delta \circ \gamma_i(g) \leq \sigma_{i\infty}(g)$.

Now, let us prove that $\sigma_{i\infty}(g) \leq \delta \circ \gamma_i(g)$:
By Corollary 6.3.4, we know that $(g_{\ll,n_j})_{j>i,n_g}$ is a $\ll$-increasing sequence whose supremum is g. By the same argument, we have that $((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}})_{j'>j+1,n_j}$ is a $\ll$-increasing sequence whose supremum is $\sigma_{ij+1}(g_{\ll,n_j})$. Fix j be arbitrary big. In particular $j > i+2, n_g+2$. There exists $j' > j+1, n_j$ such that:

$$\sigma_{ij+1}(g_{\ll,n_{j-2}}) \ll \sigma_{ij+1}(g_{\ll,n_{j-1}}) \ll (\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}} \text{ in } S_{j+1}.$$

Furthermore, by hypothesis (ii), observe that $\sigma_{ij+1}(g_{\ll,n_{j-2}}) \ll \sigma_{ij+1}(g_{\ll,n_{j-1}})$ in $\chi_{n_{j+1}} \subseteq S_{j+1}$. Using hypothesis (i) on the first comparison and that τ_{j+1j+2} is a Cu-morphism on the second comparison, we deduce the following:

$$\sigma_{j+1j+2}\circ\sigma_{ij+1}(g_{\ll,n_{j-2}}) \ll \tau_{j+1j+2}\circ\sigma_{ij+1}(g_{\ll,n_{j-1}}) \ll \tau_{j+1j+2}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \text{ in } S_{j+2}.$$

Again, by hypothesis (ii), observe that $\sigma_{j+1j+2}\circ\sigma_{ij+1}(g_{\ll,n_{j-2}}), \tau_{j+1j+2}\circ\sigma_{ij+1}(g_{\ll,n_{j-1}}) \in \chi_{n_{j+2}} \subseteq S_{n_{j+2}} = T_{n_{j+2}}$. Hence, hypothesis (i) still applies here, and repeating this process we obtain for any $j' \geq k' > j+1, n_j$:

$$\sigma_{j+1j'+1}\circ\sigma_{ij+1}(g_{\ll,n_{j-2}}) \ll \tau_{j+1j'+1}\circ\sigma_{ij+1}(g_{\ll,n_{j-1}}) \ll \tau_{j+1j'+1}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \text{ in } S_{j'+1}.$$

Composing with $\sigma_{j'+1\infty}$, we obtain for any $j' \geq k' > j+1, n_j$:

$$\sigma_{j'+1\infty}\circ\sigma_{j+1j'+1}\circ\sigma_{ij+1}(g_{\ll,n_{j-2}}) \ll \sigma_{j'+1\infty}\circ\tau_{j+1j'+1}((\sigma_{ij+1}(g_{\ll,n_j}))_{\ll,n_{j'}}) \text{ in } S.$$

Now, taking suprema over j' first and then over j, we conclude that $\sigma_{i\infty}(g) \leq \delta\circ\gamma_i(g)$.

Putting everything together, we conclude that for any $i \in \mathbb{N}$ and any $s \in S_i$, we have $\delta\circ\gamma_i(s) = \sigma_{i\infty}(s)$. It follows that $\delta\circ\gamma = \mathrm{id}_S$. Symmetrically, we have $\gamma\circ\delta = \mathrm{id}_T$. Now, since any PoM-isomorphism between two Cu-semigroups is in fact a Cu-isomorphism (see Lemma 3.1.14), we conclude $S \simeq T$ as Cu-semigroups through γ and $\delta = \gamma^{-1}$. □

6.4 The Evans-Kishimoto construction

6.4.1. In this last section, we study the Cu-semigroup of a specific construction of NCCW 1 algebras: the Evans-Kishimoto folding interval algebras. We refer the reader to [31] for more details. We point out that these C^*-algebras are a generalization of Elliott-Thomsen dimension-drop interval algebras (see Paragraph 5.2.2) even though they were considered beforehand.

By Proposition 4.4.4, we know how to compute the Cu-semigroup of such a construction. In fact, we will check that we can picture it as a Cu-subsemigroup of $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ and adapt all the above to these Cu-semigroups. Furthermore, we will generalize Theorem 6.3.8 to inductive systems of finite sums of both Lsc and Cu-semigroups obtained from the Evans-Kishimoto construction.

6.4.2. *Evans-Kishimoto folding interval algebras*:

These C^*-algebras were first considered in [31, Lemma 2.1 (2.14)] and they are constructed as follows: Let q be a natural number and denote the full matrix algebra of size q by M_q. Let e and $f := (1_{M_q} - e)$ be two (non-trivial) projections in M_q. For any $n \in \mathbb{N}$, we consider the following pullback:

$$\begin{array}{ccc} \mathcal{I}^n_{q,e} & \xrightarrow{\pi_1} & C([0,1], \overset{n}{\underset{1}{\otimes}} M_q) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle (ev_0, ev_1)} \\ (\overset{n-1}{\underset{1}{\otimes}} M_q) \oplus (\overset{n-1}{\underset{1}{\otimes}} M_q) & \xrightarrow[(i^n_0, i^n_1)]{} & (\overset{n}{\underset{1}{\otimes}} M_q) \oplus (\overset{n}{\underset{1}{\otimes}} M_q) \end{array}$$

where $i^n_0, i^n_1 : (\overset{n-1}{\underset{1}{\otimes}} M_q) \oplus (\overset{n-1}{\underset{1}{\otimes}} M_q) \longrightarrow \overset{n}{\underset{1}{\otimes}} M_q$ are injections that are constructed in [31, Lemma 2.1 (2.11)/(2.12)].

In other words, $\mathcal{I}^n_{q,e} := A((\overset{n-1}{\underset{1}{\otimes}} M_q) \oplus (\overset{n-1}{\underset{1}{\otimes}} M_q), \overset{n}{\underset{1}{\otimes}} M_q, i^n_0, i^n_1)$. For $n = 1$, observe that $\mathcal{I}^1_{q,1_{M_q}} = I_q$ is the Elliott-Thomsen dimension-drop interval algebra (see Paragraph 5.2.2). By Proposition 4.4.4, we know that:

$$\begin{aligned} &\mathrm{K}_0(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z} & \mathrm{Cu}(\mathcal{I}^n_{q,e}) &\simeq \{f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \mid f(0), f(1) \in q\overline{\mathbb{N}}\} \\ &\mathrm{K}_1(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z}/q\mathbb{Z} & &\simeq \{f \in \mathrm{Lsc}([0,1], \tfrac{1}{q^n}\overline{\mathbb{N}}) \mid f(0), f(1) \in \tfrac{q}{q^n}\overline{\mathbb{N}}\} \end{aligned}$$

First, we observe that the projection e does not play any role at the level of positive elements, so we usually omit it in notations when dealing with the Cu-semigroup. Now, we have seen that we can picture $\mathrm{Cu}(\mathcal{I}^n_q) \simeq \{f \in \mathrm{Lsc}([0,1], \frac{1}{q^n}\overline{\mathbb{N}}) \mid f(0), f(1) \in \frac{q}{q^n}\overline{\mathbb{N}}\}$ as a Cu-subsemigroup of $\mathrm{Lsc}([0,1], \frac{1}{q^n}\overline{\mathbb{N}})$.

The next step is to find adequate substitutes of Γ_m, χ_m for $\mathrm{Cu}(\mathcal{I}^n_q)$ that will allow us to adapt all the results of the previous sections to $\mathrm{Cu}(\mathcal{I}^n_q)$.

Definition 6.4.3. Let q, n be fixed natural numbers as before. Let $m \in \mathbb{N}$ and let $\{U_k\}_{k=1}^{3^m}$ be the canonical 1-thin cover of $[0,1]$ of size $1/3^m$.

We define $\chi_m(\mathrm{Cu}(\mathcal{I}^n_q)) := \chi_m([0,1], \frac{1}{q^n}\overline{\mathbb{N}}) \bigcap \mathrm{Cu}(\mathcal{I}^n_q)$. We also define $\Gamma_m(\mathrm{Cu}(\mathcal{I}^n_q))$ to be the set of m-piecewise characteristic functions of $\mathrm{Lsc}([0,1], \frac{1}{q^n}\overline{\mathbb{N}})$ of the following form:

$$\begin{aligned} g : [0,1] &\longrightarrow \tfrac{1}{q^n}\overline{\mathbb{N}} \\ x &\longmapsto \begin{cases} s_1 \in \{0, \frac{q}{q^n}\}, \text{ if } x \in [0, 1/3^m[, \\ s_k \in \{\frac{j}{q^n}\}_{0 \leq j \leq q}, \text{ if } x \in U_k, \text{ where } 2 \leq k \leq 3^m - 1, \\ s_{3^m} \in \{0, \frac{q}{q^n}\}, \text{ if } x \in](3^m - 1)/3^m, 1]. \end{cases} \end{aligned}$$

In other words, $\Gamma_m((\mathrm{Cu}(\mathcal{I}^n_q)) := \{g \in \chi_m([0,1], \frac{1}{q^n}\overline{\mathbb{N}}) \mid s_1, s_{3^m} \in \{0, \frac{q}{q^n}\}, s_k \in \{\frac{j}{q^n}\}_{0 \leq j \leq q}\}$. Notice that the values $s_1, ..., s_{3^m}$ and $\{g(x), x \in X \setminus (\bigcup_{k=1}^{3^n} U_k)\}$ are such that the resulting function g is lower-semicontinuous.

When q is fixed and no confusion can be made, we sometimes write $\Gamma_m^n := \Gamma_m(\mathrm{Cu}(\mathcal{I}_q^n))$ and $\chi_m^n := \chi_m(\mathrm{Cu}(\mathcal{I}_q^n))$. In the end, for a fixed q, one can observe that $\Gamma_m^n = \{g \in \chi_m^n \mid g \leq \frac{q}{q^n}.1_{[0,1]}\}$.

Proposition 6.4.4. *Let q, n, m be fixed natural numbers. Then χ_m^n and $\bigcup_{m\in\mathbb{N}} \chi_m^n$ are* PoM.

Proof. Since $\chi_m^n := \{g \in \chi_m([0,1], \frac{1}{q^n}\overline{\mathbb{N}}) \mid s_1, s_{3^m} \in \frac{q}{q^n}\overline{\mathbb{N}}\}$ we trivially deduce that it is a submonoid of χ_m^n. So it is clear that χ_m^n is a PoM and also we still have $\chi_m^n \subset \chi_{m'}^n$ for any $m \leq m'$. Thus, we conclude that $\bigcup_{m\in\mathbb{N}} \chi_m^n$ is a PoM. □

Remark 6.4.5. Let q, n, m be fixed natural numbers. We observe that for any $g \in \chi_m^n$, its canonical decomposition $\sum_{k=0}^{\infty} \frac{1}{q^n}.1_{V_k}$ (see Lemma 6.1.14) can be written as $\sum_{k=0}^{\infty}(\sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i})$. Since $g \in \chi_m^n$ we know that its canonical decomposition is a finite sum and that the sequence $(V_k)_k$ is $\subseteq$-decreasing. On the other hand $g(0), g(1) \in \frac{q}{q^n}\mathbb{N}$, and one can check that $(\sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i})_{k\in\mathbb{N}}$ is a decreasing sequence in Γ_m^n. So the canonical decomposition of $g \in \chi_m^n$ can be seen as a sum of elements of a decreasing sequence in $\Gamma_m^n \subseteq \Gamma_m([0,1], \overline{\mathbb{N}})$. Again, we may say that Γ_m^n generates χ_m^n; see Remark 6.1.16.

In fact, an analogous version of Lemma 6.1.15 holds for $\mathrm{Cu}(\mathcal{I}_q^n)$, using χ_m^n, Γ_m^n. In particular, for any two $g, g'' \in \chi_m^n$ such that $g'' \ll g$ and any $m' > m$, we can find $g' \in \chi_{m'}^n$ such that $g'' \ll g' \ll g$ in $\chi_{m'}^n$.

Proposition 6.4.6. *Let q, n, m be fixed natural numbers. Then $\bigcup_{m\in\mathbb{N}} \chi_m^n$ is dense in* $\mathrm{Cu}(\mathcal{I}_q^n)$.

Proof. Let $f \in \mathrm{Cu}(\mathcal{I}_q^n)$. First, observe that for any $g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ such that $g \ll f$, we can find $f' \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ such that $g \ll f' \ll f \leq \infty$. So without loss of generality, we can suppose that $f \ll \infty$. Equivalently, the canonical decomposition of f is a finite sum.
Thus, by Corollary 6.3.4 we know that $f = \sup_j f_{\ll,j}$. On the other hand, since $f(0), f(1)$ are compact, there exists some $l \in \mathbb{N}$ such that for any $j \geq l$, then $f_{\ll,j}(0) = f(0), f_{\ll,j}(1) = f(1)$. We obtain that $(f_{\ll,j})_{j\geq l}$ is a $\ll$-increasing sequence in $\mathrm{Cu}(\mathcal{I}_q^n)$ whose supremum is f. The result follows. □

6.4.7. The above corollary allows us to adapt the way we approximate elements of $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ using Γ_m, χ_m to $\mathrm{Cu}(\mathcal{I}_q^n)$ using Γ_m^n, χ_m^n. A fortiori, it allows us to define a discrete Cu-semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Cu}(\mathcal{I}_q^n), T)$ as follows:

Definition 6.4.8. Let T be a Cu-semigroup and let $\alpha, \beta : \mathrm{Cu}(\mathcal{I}_q^n) \longrightarrow T$ be two Cu-morphisms. We define a discrete semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Cu}(\mathcal{I}_q^n), T)$ by:

$$dd_{\mathrm{Cu}}(\alpha,\beta) := \inf_{m\in\mathbb{N}}\{1/3^m \mid \forall g' \ll g \in \Gamma_m^n, \alpha(g') \leq \beta(g) \text{ and } \beta(g') \leq \alpha(g)\}.$$

If the infimum defined does not exist, we set the value to ∞.

Remark 6.4.9. As in Lemma 6.2.9, one can easily show that $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^m$ if and only if $\alpha \underset{\Gamma_m^n}{\simeq} \beta$. In fact, all of the statements of Lemma 6.2.9 nicely adapt in this context.

Proposition 6.4.10. *Let $\alpha, \beta : \mathrm{Cu}(\mathcal{I}_q^n) \longrightarrow T$ be* Cu*-morphisms. The following are equivalent:*
(i) $dd_{\mathrm{Cu}}(\alpha,\beta) = 0$.
(ii) $\alpha \underset{\Gamma_m^n}{\simeq} \beta$, for any $m \in \mathbb{N}$.
(ii') $\alpha \underset{\Gamma_m^n}{\approx} \beta$, for any $m \in \mathbb{N}$.
(iii) $\alpha = \beta$.
Hence dd_{Cu} is a semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(\mathrm{Cu}(\mathcal{I}_q^n), T)$.

Proof. As mentioned in Remark 6.4.9, one can easily obtain (i) is equivalent to (ii) is equivalent to (ii'), similarly as before. (iii) implies (i) is trivial. Now suppose that $dd_{\mathrm{Cu}}(\alpha,\beta) = 0$. Let $f \in \mathrm{Cu}(\mathcal{I}_q^n)$. As observed in Remark 6.4.5, we write $f = \sum_{k=0}^{\infty}(\sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i})$. Also, for each $k \in \mathbb{N}$ we can find a $\ll$-increasing sequence $(g_{k,j})_j$ in $\bigcup_{m\in\mathbb{N}} \chi_m^n$ such that $\sup_{j\in\mathbb{N}} g_{k,j} = \sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i}$. In fact, this sequence belongs to $\bigcup_{m\in\mathbb{N}} \Gamma_m^n$.
Now by hypothesis, we know that $\alpha(g_{k,j}) \leq \beta(g_{k,j'}) \leq \beta(\sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i})$ and $\beta(g_{k,j}) \leq \alpha(g_{k,j'}) \leq \alpha(\sum_{i=kq}^{(k+1)q-1} \frac{1}{q^n}.1_{V_i})$ for any $j < j'$. Finally, passing to suprema, we obtain that $\alpha(f) \leq \beta(f)$ and $\beta(f) \leq \alpha(f)$, which ends the proof. □

Lemma 6.4.11. *Let q, n, m be fixed natural numbers and let $f \in \mathrm{Cu}(\mathcal{I}_q^n)$ such that $f \ll \infty$. Then $\chi_{m,f}^n$ has a largest element that we write $f_{\ll,m}^n$. In fact, $f_{\ll,m}^n \leq f_{\ll,m} \ll f$ and analogously, we also get that $(f_{m,\ll}^n)_{m\in\mathbb{N}}$ is a $\ll$-increasing sequence whose supremum is f. See Corollary 6.3.4.*

Proof. Let q, n, m be fixed natural numbers and let $f \in \mathrm{Cu}(\mathcal{I}_q^n)$ such that $f \ll \infty$. Consider $f_{\ll,m}$ as in Corollary 6.3.4. Then $f_{\ll,m}^n$ is constructed as follows:
(1) Put $f_{\ll,m|U_1}^n := \max_{\frac{q}{q^n}\mathbb{N}}\{k \leq f_{\ll,m|U_1}\}$ and $f_{\ll,m|U_{3^m-1}}^n := \max_{\frac{q}{q^n}\mathbb{N}}\{k \leq f_{\ll,m|U_{3^m-1}}\}$.

(2) Put $f^n_{\ll,m|[0,1]\backslash(U_1\cup U_{3^m-1})} := f_{\ll,m|[0,1]\backslash(U_1\cup U_{3^m-1})}$.
Obviously, $f^n_{\ll,m} \leq f_{\ll,m} \ll f$. The last statement of the lemma follows from density of $\bigcup\limits_{m\in\mathbb{N}} \chi^n_m$, as before. □

Remark 6.4.12. Observe that we shown a similar statement in the proof of Proposition 6.4.6. Also, an analogous version of Corollary 6.3.6 holds for $\mathrm{Cu}(\mathcal{I}^n_q)$. That is, $(.)^n_{\ll,m}$ is compatible with $+, \leq, \ll$ and suprema of $\ll$-increasing sequences in $\bigcup\limits_{l\in\mathbb{N}} \chi^n_l$.

6.4.13. In the end, we have defined analogous tools $\Gamma^n_m, \chi^n_m, f^n_{\ll,m}$ for $\mathrm{Cu}(\mathcal{I}^n_q)$ that act the same and satisfy the same properties as before. In particular, we can now adapt the proof of Theorem 6.3.8 to this new context and everything is proved similarly:

Theorem 6.4.14. *Let $(q_i)_{i\in\mathbb{N}}, (n_i)_{i\in\mathbb{N}}$ be two sequences of natural numbers. For any $i \in \mathbb{N}$, consider* Cu*-morphisms $\sigma_{ii+1}, \tau_{ii+1} : \mathrm{Cu}(\mathcal{I}^{n_i}_{q_i}) \longrightarrow \mathrm{Cu}(\mathcal{I}^{n_{i+1}}_{q_{i+1}})$ that define two inductive sequences $(\mathrm{Cu}(\mathcal{I}^{n_i}_{q_i}), \sigma_{ij})_{i\in\mathbb{N}}$ and $(\mathrm{Cu}(\mathcal{I}^{n_i}_{q_j}), \tau_{ij})_{i\in\mathbb{N}}$ in* Cu*. Let S and T be their respective inductive limits. Suppose there exists a strictly increasing sequence of positive integers $(m_i)_i$ such that:*
(i) $\sigma_{ii+1} \underset{\Gamma^{n_i}_{m_i}}{\approx} \tau_{ii+1}$, for any $i \in \mathbb{N}$.
(ii) For any $i \leq j$ and any $l \in \mathbb{N}$, we have $\sigma_{ij}(\chi^{n_i}_l), \tau_{ij}(\chi^{n_i}_l) \subseteq \chi^{n_{i+1}}_l$, where $\sigma_{ij} := \sigma_{j-1j} \circ ... \circ \sigma_{ii+1}$ and $\tau_{ij} := \tau_{j-1j} \circ ... \circ \tau_{ii+1}$.
Then, $S \simeq T$ as Cu*-semigroups.*

Proof. Combine Remark 6.4.9 and Theorem 6.4.14 to get the result. □

6.4.15. *A generalization to finite direct sums*:
We end this chapter by extending the approximate intertwining theorem to a more general case: The inductive sequence is now a sequence of finite direct sums of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $\mathrm{Cu}(\mathcal{I}^n_q)$, where X is the circle or the interval and q, n are natural numbers. As done for $\mathrm{Cu}(\mathcal{I}^n_q)$, the aim is to find adequate $\Gamma_m, \chi_m, f_{\ll,m}$ for the finite direct sums that will allow us to adapt all the results of the previous sections to this generalization. This notion will be extended component-wise.

Definition 6.4.16. Let $S := \overset{l}{\underset{k=1}{\oplus}} S_k$, where S_k is either $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ or $\mathrm{Cu}(\mathcal{I}^n_q)$ for some $q, n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and let $f \in S$ such that $f \ll \infty$. We write $f = (f_1, ..., f_l)$ and we define:

$$\Gamma_m(S) := \overset{l}{\underset{k=1}{\oplus}} \Gamma_m(S_k).$$
$$\chi_m(S) := \overset{l}{\underset{k=1}{\oplus}} \chi_m(S_k).$$
$$f_{\ll,m} := (f^*_{1\,\ll,m}, ..., f^*_{l\,\ll,m}).$$

where $f^*_{k\ll,m} := f_{k\ll,m}$ if $S_k \simeq \mathrm{Lsc}(X,\overline{\mathbb{N}})$ and $f^*_{k\ll,m} := f^n_{k\ll,m}$ if $S_k \simeq \mathrm{Cu}(\mathcal{I}^n_q)$.
Now let T be a Cu-semigroup. For any two Cu-morphisms $\alpha,\beta : S \longrightarrow T$, we define a discrete semimetric on $\mathrm{Hom}_{\mathrm{Cu}}(S,T)$ by:

$$dd_{\mathrm{Cu}}(\alpha,\beta) := \inf_{m\in\mathbb{N}}\{1/3^m, \forall g' \ll g \in \Gamma_m, \alpha(g') \leq \beta(g) \text{ and } \beta(g') \leq \alpha(g)\}.$$

If the infimum defined does not exist, we set the value to ∞.

Lemma 6.4.17. *Let $S := \bigoplus_{k=1}^{l} S_k$, where S_k is either $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$, $\mathrm{Lsc}(\mathbb{T},\overline{\mathbb{N}})$ or $\mathrm{Cu}(\mathcal{I}^n_q)$ for some $q,n \in \mathbb{N}$. Let T be a* Cu*-semigroup. Then, for any two* Cu*-morphisms $\alpha,\beta : S \longrightarrow T$ we have $dd_{\mathrm{Cu}}(\alpha,\beta) = \max_{k=1,...,l}(dd_{\mathrm{Cu}}(\alpha_k,\beta_k))$, where $\alpha_k,\beta_k : S_k \overset{i_k}{\hookrightarrow} S \longrightarrow T$ are* Cu*-morphisms defining α,β.*
If moreover $T := \bigoplus_{k=1}^{l'} T_k$, then we can picture α,β as:

$$\alpha := \begin{pmatrix} \alpha_{11} & ... & \alpha_{l1} \\ . & & . \\ . & & . \\ \alpha_{1l'} & ... & \alpha_{ll'} \end{pmatrix} \qquad \beta := \begin{pmatrix} \beta_{11} & ... & \beta_{l1} \\ . & & . \\ . & & . \\ \beta_{1l'} & ... & \beta_{ll'} \end{pmatrix}$$

where $\alpha_{kk'} : S_k \overset{i_k}{\hookrightarrow} S \overset{\alpha}{\longrightarrow} T \overset{\pi_{k'}}{\longrightarrow} T_{k'}$ are Cu*-morphisms defining α,β (respectively $\beta_{kk'}$). Then $dd_{\mathrm{Cu}}(\alpha,\beta) = \max_{k,k'} dd_{\mathrm{Cu}}(\alpha_{kk'},\beta_{kk'})$.*

Proof. Let us prove the second statement and the first result can be proved similarly (by taking $l' = 1$ and any $T \in \mathrm{Cu}$). Let $S := \bigoplus_{k=1}^{l} S_k$ and $T := \bigoplus_{k=1}^{l'} T_k$ be Cu-semigroups as in the theorem. Let $\alpha,\beta : S \longrightarrow T$ be Cu-morphisms. We are going to show that $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^m$ if and only if $\max_{k,k'} dd_{\mathrm{Cu}}(\alpha_{kk'},\beta_{kk'}) \leq 1/3^m$.
Suppose that $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^m$. Let k,k' be indices of the direct sums. Then for any $g',g \in \Gamma_m(S_k)$ such that $g' \ll g$, then $i_k(g') \ll i_k(g)$ in $\Gamma_m(S)$. Using the hypothesis, we obtain that $\alpha \circ i_k(g') \leq \beta \circ i_k(g)$ and $\beta \circ i_k(g') \leq \alpha \circ i_k(g)$. In particular, projecting onto $T_{k'}$, we get $\pi_{k'} \circ \alpha \circ i_k(g') \leq \pi_{k'} \circ \beta \circ i_k(g)$ and $\pi_{k'} \circ \beta \circ i_k(g') \leq \pi_{k'} \circ \alpha \circ i_k(g)$. We conclude that $dd_{\mathrm{Cu}}(\alpha_{kk'},\beta_{kk'}) \leq 1/3^m$ for any k,k'.
Conversely, suppose that $\max_{k,k'} dd_{\mathrm{Cu}}(\alpha_{kk'},\beta_{kk'}) \leq 1/3^m$. Let $g',g \in \Gamma_m(S)$ such that $g' \ll g$. Then $g = (g_1,...,g_l), g' = (g'_1,...,g'_l)$ for some $g_k, g'_k \in S_k$ such that $g_k \ll g'_k$. Now we compute $\alpha(g') = (\sum_{k=1}^{l}\pi_1 \circ \alpha \circ i_k(g'_k),..., \sum_{k=1}^{l}\pi_{l'} \circ \alpha \circ i_k(g'_k)) \ll (\sum_{k=1}^{l}\pi_1 \circ \beta \circ i_k(g_k),..., \sum_{k=1}^{l}\pi_{l'} \circ \beta \circ i_k(g_k)) = \beta(g)$.
We conclude that $\alpha(g') \leq \beta(g)$ and $\beta(g') \leq \alpha(g)$, which ends the proof. □

6.4.18. Now that everything is defined component-wise it is almost immediate that all of the above results apply:
In particular, χ_m and $\bigcup_{m\in\mathbb{N}} \chi_m$ are PoM. The former is generated by Γ_n and the latter is dense in S. It is not hard to check that the results in Section 6.1 adapt to this component-wise definition. Also, as a consequence of Lemma 6.4.17, dd_{Cu} is a semimetric and one can check that the results in Section 6.2 adapt to this component-wise definition. Finally, $(.)_{\ll,m}$ is again compatible with $+, \leq, \ll$ and suprema of $\ll$-increasing sequences in $\bigcup_{l\in\mathbb{N}} \chi_l^n$.

In the end, we have defined analogous tools $\Gamma_m, \chi_m, f_{\ll,m}$ for S being a finite direct sum of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $\mathrm{Cu}(\mathcal{I}_q^n)$, where X is the circle or the interval and q, n are natural numbers, that act the same and satisfy the same properties as before. In particular, we can now adapt the proof of Theorem 6.3.8 to this new context and everything works out the same:

Theorem 6.4.19. *Let us consider two inductive sequences* $(S_i, \sigma_{i,j})_{i\in\mathbb{N}})$ *and* $(S_i, \tau_{i,j})_{i\in\mathbb{N}})$ *in* Cu*, where* $S_i := \bigoplus_{k=0}^{l_i} S_{k,i}$ *for some* $S_{k,i}$ *that are either* $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$*,* $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ *or* $\mathrm{Cu}(\mathcal{I}_{q_{i,k}}^{n_{i,k}})$ *for some* $q_{i,k}, n_{i,k} \in \mathbb{N}$*. Let* S *and* T *be their respective inductive limits. Suppose there exists a strictly increasing sequence of positive integers* $(m_i)_i$ *such that:*
(i) $\sigma_{ii+1} \underset{\Gamma_{m_i}(S_i)}{\approx} \tau_{ii+1}$*, for any* $i \in \mathbb{N}$*.*
(ii) For any $i \leq j$ *and any* $l \in \mathbb{N}$*, we have* $\sigma_{ij}(\chi_l(S_i)), \tau_{ij}(\chi_l(S_i)) \subseteq \chi_l(S_j)$*, where* $\sigma_{ij} := \sigma_{j-1j} \circ ... \circ \sigma_{ii+1}$ *and* $\tau_{ij} := \tau_{j-1j} \circ ... \circ \tau_{ii+1}$*.*
Then, $S \simeq T$ *as* Cu*-semigroups.*

6.4.20. (Open line of research)
Since we have been adapting our approximate intertwining theorem to different settings, it seems that we may be able to generalize the constructions done in this chapter as follows:
Let S be a Cu-semigroup that admits a countable number of subsets χ_n such that $\chi_n \in$ PoM and such that $\bigcup_{n\in\mathbb{N}} \chi_n \in$ PoM is dense in S. Set $S_{\ll\infty} := \{s \in S \mid s \ll \infty\}$. Suppose that there exist $\epsilon_n : S_{\ll\infty} \longrightarrow \chi_n$ such that for any $s \in S_{\ll\infty}$, we have $\epsilon_n(f) \ll \epsilon_{n+1}(f)$ and $\sup_n \epsilon_n(f) = f$. This would allow us to create a semimetric dd_{Cu} and we might be able to generalize the approximate intertwining theorem to any inductive sequence of Cu-semigroups satisfying all the above.

Chapter 7

A concrete use of Cu_1 in the classification of certain NCCW 1 algebras

In this chapter, we exhibit a concrete example of two C^*-algebras A and B with the following features: these are non simple, unital, separable NCCW 1 algebras. Moreover, they have isomorphic K_1 group, with torsion, and isomorphic Cu-semigroup. However, they are not isomorphic as their Cu_1-semigroup distinguish them.
This example has been greatly inspired by the lines of work done in [34], as the structure of these algebras are in many ways similar, even though the building blocks, properties and arguments involved are quite different. Let us first recall a construction done in [31].

7.1 Preliminaries

7.1.1. [31, Section 2] Recall the construction of Evans-Kishimito folding interval algebras introduced in Paragraph 6.4.2. Let q be a fixed natural number. Let $n \in \mathbb{N}$ and e be a projection of M_q and consider $\mathcal{I}^n_{q,e}$. By Proposition 4.4.4, we know that:

$$\mathrm{K}_0(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z} \qquad\qquad \mathrm{K}_1(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z}/q\mathbb{Z}$$

Arguing similarly as in [31, Proof of 2.2], we now consider the two following paths in $[0,1]$, $\xi_0 : t \longmapsto t/2$ and $\xi_1 : t \longmapsto 1 - t/2$. We have the following *-homomorphism:

$$\begin{aligned} \psi_{n,e} : \mathcal{I}^n_{q,e} &\longrightarrow \mathcal{I}^{n+1}_{q,e} \\ f &\longmapsto f(\xi_0) \otimes e + f(\xi_1) \otimes f \end{aligned}$$

Moreover,

$$\mathrm{K}_0(\psi_{n,e}) : \mathbb{Z} \xrightarrow{\times q} \mathbb{Z} \qquad\qquad \mathrm{K}_1(\psi_{n,e}) : \mathbb{Z}/q\mathbb{Z} \xrightarrow{\times \operatorname{rank}(e)} \mathbb{Z}/q\mathbb{Z}.$$

Lemma 7.1.2. *Let us end up these preliminaries with a lemma on inductive limits in AbGroups that will be helpful later on:*
Let p_1, p_2 be distinct prime numbers. Then $\varinjlim(\mathbb{Z}/p_1p_2\mathbb{Z} \xrightarrow{\times p_1} \mathbb{Z}/p_1p_2\mathbb{Z} \xrightarrow{\times p_1} ...) \simeq \mathbb{Z}/p_2\mathbb{Z}$.

Proof. Let p_11, p_2 be distinct prime numbers. A fortiori p_1 and p_2 are coprime. Hence, from the well-known Chinese remainder theorem, we know that $\mathbb{Z}/p_1p_2\mathbb{Z} \simeq \mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z}$. Also, $\mathbb{Z}/p_2\mathbb{Z} \xrightarrow{\times p_1} \mathbb{Z}/p_2\mathbb{Z}$ is an isomorphism of abelian groups -it corresponds to a permutation in $\mathbb{Z}/p_2\mathbb{Z}$-, and $\mathbb{Z}/p_1\mathbb{Z} \xrightarrow{\times p_1} \mathbb{Z}/p_1\mathbb{Z}$ is the zero morphism. Let us write $\varphi_{p_1} : \mathbb{Z}/p_1p_2\mathbb{Z} \xrightarrow{\times p_1} \mathbb{Z}/p_1p_2\mathbb{Z}$, we have hence $\operatorname{im}\varphi_{p_1} \simeq \mathbb{Z}/p_1\mathbb{Z}$ and $\ker\varphi_{p_1} \simeq \mathbb{Z}/p_2\mathbb{Z}$. In the end, the inductive system considered is naturally isomorphic to $(\mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \xrightarrow{0\times\varphi_{p_1}} \mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \xrightarrow{0\times\varphi_{p_1}} ...)$ and we deduce the result. □

7.2 The example

7.2.1. We will now construct two C^*-algebras A and B that are inductive limits of direct sums of building blocks that are either matrix algebras over $C([0,1])$ or matrix algebras over some Evans-Kishimito folding interval algebra $\mathcal{I}^n_{q,e}$. As stated before, these NCCW 1 algebras are non simple, unital, separable and they have stable rank one. We will also prove that $\mathrm{K}_1(A) \simeq \mathrm{K}_1(B)$ and $\mathrm{Cu}(A) \simeq \mathrm{Cu}(B)$, whereas $\mathrm{Cu}_1(A) \not\simeq \mathrm{Cu}_1(B)$.

Definition 7.2.2. Let us use the following notations:
• $(t_k)_k$ is a countable dense subset of $[0,1]$,
• $(p_k)_k$ denotes the prime numbers ($p_0 = 2$ and convention $p_{-1} = 1$),
• $(q_k)_k$ is defined by $q_0 = p_0$ and $q_k := p_kp_{k-1}$ for $k \geq 1$,
• $(m_k)_k$ is a strictly increasing sequence of natural numbers such that $1/q_0^{m_k-3} \leq 1/3^k$ for any $k \in \mathbb{N}$.
Also, for any $k \in \mathbb{N}$ let e_A^k, e_B^k be projections of M_{q_k} such that $\operatorname{rank}(e_A^k) = p_{k-1}$ and $\operatorname{rank}(e_B^k) = p_k$. Finally, we write

$$\begin{cases} \mathcal{I}^n_{q_k} := \mathcal{I}^n_{q_k,e_A^k} \\ \mathcal{J}^n_{q_k} := \mathcal{I}^n_{q_k,e_B^k} \end{cases}$$

and we do not refer to e_A^k, e_B^k no more for notational purposes.
Let us build the inductive systems of A and B. Consider

$$A_0 = C[0,1],$$
$$A_1 = M_{q_0^{m_0}}(\mathcal{I}^1_{q_0}) \oplus C[0,1],$$
$$A_2 = M_{q_0^{m_0}q_0^{m_1}}(\mathcal{I}^2_{q_0}) \oplus M_{q_1^{m_1}}(\mathcal{I}^1_{q_1}) \oplus C[0,1],$$
$$...$$
$$A_n = M_{q_0^{m_0}...q_0^{m_{n-1}}}(\mathcal{I}^n_{q_0}) \oplus M_{q_1^{m_1}...q_1^{m_{n-1}}}(\mathcal{I}^{n-1}_{q_1}) \oplus ... \oplus M_{q_i^{m_i}...q_i^{m_{n-1}}}(\mathcal{I}^{n-i}_{q_i}) \oplus ... \oplus M_{q_{n-1}^{m_{n-1}}}(\mathcal{I}^1_{q_{n-1}}) \oplus C[0,1],$$

Let us simplify notations by writing $[n,i] := \prod_{j=i}^{n-1} q_i^{m_j}$, for any $0 \leq i \leq n-1$, and $[n,n] := 1$. Notice that $[n+1,i] = q_i^{m_n}[n,i]$ for any $0 \leq i \leq n-1$. Thus, we finally rewrite:

$$\begin{cases} A_n := \bigoplus_{i=0}^{n-1} M_{[n,i]}(\mathcal{I}^{n-i}_{q_i}) \oplus M_{[n,n]}(C[0,1]) \\ B_n := \bigoplus_{i=0}^{n-1} M_{[n,i]}(\mathcal{J}^{n-i}_{q_i}) \oplus M_{[n,n]}(C[0,1]) \end{cases}$$

Now that we have the algebras, let us build the morphisms of our inductive sequences. Let $n \in \mathbb{N}$. First, we consider the following partial morphisms:
(i) For any $0 \leq i \leq n-1$ we define:

$$\phi^i_{nn+1} : M_{[n,i]}(\mathcal{I}^{n-i}_{q_i}) \longrightarrow M_{[n+1,i]}(\mathcal{I}^{n-i+1}_{q_i})$$

$$f \longmapsto \begin{pmatrix} f(\xi_0) \otimes e_A^i + f(\xi_1) \otimes f_A^i & & & \\ & f(1/r) \otimes 1_{M_{q_i}} & & \\ & & ... & \\ & & & f(r-1/r) \otimes 1_{M_{q_i}} \end{pmatrix}$$

$$\psi^i_{nn+1} : M_{[n,i]}(\mathcal{J}^{n-i}_{q_i}) \longrightarrow M_{[n+1,i]}(\mathcal{J}^{n-i+1}_{q_i})$$

$$f \longmapsto \begin{pmatrix} f(\xi_0) \otimes e_B^i + f(\xi_1) \otimes f_B^i & & & \\ & f(1/r) \otimes 1_{M_{q_i}} & & \\ & & ... & \\ & & & f(r-1/r) \otimes 1_{M_{q_i}} \end{pmatrix}$$

where $r_{n,i} := q_i^{m_n}$.
(i) For $i = n$ we define:

$$\phi^n_{nn+1} : C[0,1] \longrightarrow M_{q_n^{m_n}}(\mathcal{I}^1_{q_n}) \qquad \psi^n_{nn+1} : C[0,1] \longrightarrow M_{q_n^{m_n}}(\mathcal{J}^1_{q_n})$$
$$f \longmapsto f(0) \otimes 1_{M_{q_n^{m_n+1}}} \qquad f \longmapsto f(0) \otimes 1_{M_{q_n^{m_n+1}}}$$

(iii) For $i = n+1$ we define:

$$\phi^{n+1}_{nn+1}, \psi^{n+1}_{nn+1} : C[0,1] \longrightarrow C[0,1]$$
$$f \longmapsto f_{t_n}$$

where f_{t_n} is the evaluation map at t_n.
We now define $\phi_{nn+1} : A_n \longrightarrow A_{n+1}, \psi_{nn+1} : B_n \longrightarrow B_{n+1}$ by:

$$\phi_{nn+1} := (\phi^0_{nn+1}, ..., \phi^{n-1}_{nn+1}, (\phi^n_{nn+1}, \phi^{n+1}_{nn+1})) \qquad \psi_{nn+1} := (\psi^0_{nn+1}, ..., \psi^{n-1}_{nn+1}, (\psi^n_{nn+1}, \psi^{n+1}_{nn+1}))$$

and
$$\begin{cases} A := \varinjlim_n (A_n, \phi_{nn+1}) \\ B := \varinjlim_n (B_n, \psi_{nn+1}) \end{cases}$$

Proposition 7.2.3. *Both A and B are separable unital C^*-algebras of with stable rank one.*

Proof. Since all C^*-algebras of the inductive systems are separable and unital, plus all morphisms are unital, we easily obtain that A and B are unital separable C^*-algebras. Finally we know from Lemma 4.4.3 that they have stable rank one. □

7.2.4. We will now describe the (closed two-sided) ideals of A and B. We recall that whenever we say ideal, we mean closed two-sided ideals. Further, by explicitly describing the simple ideals of A and B, we will deduce their K-Theory. The description of the simple ideals will also be used at the end of this chapter to prove that the Cu_1-semigroup distinguish these two C^*-algebras.

Lemma 7.2.5. *Let $n \in \mathbb{N}_*$ and let $0 \leq i \leq n-1$. We consider the following direct sums of full blocks of A_n:*

$$I_{n,i} := M_{[n,i]}(\mathcal{I}^{n-i}_{q_i}) \qquad I^c_{n,i} := (\bigoplus_{j=0, j\neq i}^{n-1} M_{[n,j]}(\mathcal{I}^{n-j}_{q_j})) \oplus M_{[n,n]}(C[0,1])$$

Respectively $J_{n,i}$ and $J^c_{n,i}$ in B_n. Then:

(i) Any simple ideal of A is of the form

$$\mathfrak{i}_n := \varinjlim_{m>n}(I_{m,n}, \phi_{mm'|I_{m,n}}).$$

for some $n \in \mathbb{N}$. Respectively $\mathfrak{j}_n$ for B.
(ii) For any $n \in \mathbb{N}$, write

$$\mathfrak{a}_n := \varinjlim_{m>n}(I^c_{m,n}, \phi_{mm'|I^c_{m,n}}).$$

Respectively $\mathfrak{b}_n$ for B. Then we have $\mathfrak{a}_n \oplus \mathfrak{i}_n = A$, and $\mathfrak{b}_n \oplus \mathfrak{j}_n = B$. Moreover,

$$A/\bigoplus_{n\in\mathbb{N}} \mathfrak{i}_n \simeq B/\bigoplus_{n\in\mathbb{N}} \mathfrak{j}_n \simeq \mathbb{C}.$$

Proof. We prove the statements for A and B is proved similarly.
(i) This is inspired by [37, Lemma 1]. We first describe a sufficient condition for an ideal of inductive system to be simple. Let J be a (closed two-sided) ideal of A. Using [8, Lemma 4.5], we know that $J_0 := \bigcup_{n\in\mathbb{N}}(\phi_{n\infty}(A_n) \cap J)$ is dense in J. Hence it is enough to check that the algebraic limit J_0 is simple. Write $J_n := \phi^{-1}_{n\infty}(\{\phi_{n\infty}(A_n) \cap J\})$. Observe that for any n, J_n is an ideal of A_n and hence a C^*-algebra, and we have $J_0 = \bigcup_{n\in\mathbb{N}} \phi_{n\infty}(J_n)$. We deduce that it is enough to show that for any $n \in \mathbb{N}$, any $x \in J_n$, then there exists $m \geq n$ such that $\overline{J_m\phi_{nm}(x)J_m} = J_m$.
Let $n \in \mathbb{N}$. We want to show that $\mathfrak{i}_n$ is simple. Thus we need to show that for any $g \in I_{m,n}$ such that $g \neq 0$, then there exists $m' > m$ such that $\overline{I_{m',n}\phi_{mm'}(g)I_{m',n}} = I_{m',n}$.
Let $m > n$ and let $g \in I_{m,n}$ such that $g \neq 0$. Observe that $\{k/r_{m',n}, k = 1, ..., (r_{m',n} - 1), m' > m\}$ is dense in $[0, 1]$. Also note that if the support of g contains an interval of length s, then so does the support of $\phi_{mm'}(g)$. We deduce that there exists $m' \geq m$ such that $k/r_{m',n} \in \mathrm{supp}(\phi_{mm'}(g))$ for some $1 \leq k \leq (r_{m',n} - 1)$ and hence the ideal generated by $\phi_{mm'}(g)$ is dense in $I_{m',n}$. We conclude that $\mathfrak{i}_n$ is a simple C^*-algebra for any $n \in \mathbb{N}$.
Now, by [8, Lemma 4.5] and the above, we deduce that any ideal of A is in fact of the form $\varinjlim(I_n, \phi_{nm|I_n})$, for some n and some $I_n \in \mathrm{Lat}(A_n)$. One can check then that $\{\mathfrak{i}_n\}$ are the only simple ideals of A.
(ii) Let $n \in \mathbb{N}$. Notice that $\mathfrak{i}_n \cap \mathfrak{a}_n = \emptyset$ and that $\mathfrak{i}_n + \mathfrak{a}_n = A$ so we get the first part of the statement.
Now consider $(\mathbb{C}, e_n)_{n\in\mathbb{N}}$, where $e_n : A_n \longrightarrow \mathbb{C}$ given by $e_n(f_0, ..., f_n) = f_n(t_n)$. It is clear that $(\mathbb{C}, e_n)$ is a cocone to the inductive system (see Paragraph 1.2.3). We deduce that there exists a unique *-homomorphism $e : A \longrightarrow \mathbb{C}$ satisfying the universal properties of the direct limit.

It is trivial that e is surjective and also that $\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \subseteq \ker e$. Finally, since $\ker e$ is a closed two-sided ideal of A and also using observations made from [8, Lemma 4.5], we clearly have that $\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \supseteq \ker e$ -in fact, one can argue saying that $\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n$ is a maximal (closed two-sided) ideal of A-. We conclude that $A/\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \simeq \mathbb{C}$. □

Corollary 7.2.6.

$$\mathrm{K}_0(A) \simeq \mathrm{K}_0(B).$$
$$\mathrm{K}_1(A) \simeq \mathrm{K}_1(B).$$

Proof. Let $n \in \mathbb{N}$. By Paragraph 7.1.1 for any $m > n$, we compute that $K_0(\phi^n_{mm+1}) : \mathbb{Z} \xrightarrow{\times q_n^{m_{m+1}}} \mathbb{Z}$ and $K_1(\phi^n_{mm+1}) : \mathbb{Z}/q_n\mathbb{Z} \xrightarrow{\times p_{n-1}} \mathbb{Z}/q_n\mathbb{Z}$. Similarly, we obtain $K_0(\psi^n_{mm+1}) : \mathbb{Z} \xrightarrow{\times q_n^{m_{m+1}}} \mathbb{Z}$ and $K_1(\phi^n_{mm+1}) : \mathbb{Z}/q_n\mathbb{Z} \xrightarrow{\times p_n} \mathbb{Z}/q_n\mathbb{Z}$.

A quick computation on the one hand and using Lemma 7.1.2 on the other hand, we finally compute:

$$\mathrm{K}_0(\mathrm{i}_n) \simeq \mathrm{K}_0(\mathrm{j}_n) \simeq \mathbb{Z}[\tfrac{1}{q_n}]$$
$$\mathrm{K}_1(\mathrm{i}_n) \simeq \mathbb{Z}/p_n\mathbb{Z} \qquad \mathrm{K}_1(\mathrm{j}_n) \simeq \mathbb{Z}/p_{n-1}\mathbb{Z}$$

Note that we have fixed $p_{-1} := 1$, and hence $\mathrm{K}_1(\mathrm{j}_0) \simeq \{0\}$. Observe that for the (maximal) ideals $\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n, \bigoplus_{n\in\mathbb{N}} \mathrm{j}_n$ of A, B respectively, we have $\mathrm{K}_0(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n) \simeq \mathrm{K}_0(\bigoplus_{n\in\mathbb{N}} \mathrm{j}_n)$ and also $\mathrm{K}_1(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n) \simeq \mathrm{K}_1(\bigoplus_{n\in\mathbb{N}} \mathrm{j}_n)$ (here the isomorphism is a shift adding, or removing 0 at the beginning of the sequence).

On the other hand, since $A/\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \simeq \mathbb{C}$, we can use the 6-term exact sequence (see Theorem 1.1.13) with the canonical short-exact sequence $0 \longrightarrow \bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \longrightarrow A \longrightarrow A/\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n \longrightarrow 0$, and we obtain:

$$\begin{array}{ccccc}
\mathrm{K}_1(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n) & \longrightarrow & \mathrm{K}_1(A) & \longrightarrow & 0 \\
\delta_1 \uparrow & & & & \downarrow \\
\mathbb{Z} & \longleftarrow & \mathrm{K}_0(A) & \longleftarrow & \mathrm{K}_0(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n)
\end{array}$$

Since $\pi : A \longrightarrow A/\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n$ is unital and $\mathrm{K}_0(A/\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n)$ is generated by $[1]_{\mathbb{C}}$, we deduce that $\mathrm{K}_0(A) \longrightarrow \mathbb{Z}$ is surjective and hence $\ker(\delta_1) \simeq \mathbb{Z}$. This gives us $\delta_1 = 0$ and we deduce the following:

(*i*) $\mathrm{K}_1(A) \simeq \mathrm{K}_1(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n)$.

(*ii*) $0 \longrightarrow \mathrm{K}_0(\bigoplus_{n\in\mathbb{N}} \mathrm{i}_n) \longrightarrow \mathrm{K}_0(A) \longrightarrow \mathbb{Z} \longrightarrow 0$ is exact.

Since $\mathbb{Z}$ is free, the exact sequence of (ii) is split-exact and since all of this works out in a similar way with $\underset{n\in\mathbb{N}}{\oplus}\mathrm{j}_n$, B and we conclude the following:

$$(i)\quad \mathrm{K}_1(A) \simeq \mathrm{K}_1(\underset{n\in\mathbb{N}}{\oplus}\mathrm{i}_n) \simeq \mathrm{K}_1(\underset{n\in\mathbb{N}}{\oplus}\mathrm{j}_n) \simeq \mathrm{K}_1(B).$$
$$(ii)\quad \mathrm{K}_0(A) \simeq \mathrm{K}_0(\underset{n\in\mathbb{N}}{\oplus}\mathrm{i}_n)\oplus\mathbb{Z} \simeq \mathrm{K}_0(\underset{n\in\mathbb{N}}{\oplus}\mathrm{j}_n)\oplus\mathbb{Z} \simeq \mathrm{K}_0(B).$$

□

Lemma 7.2.7. *For any $n \in \mathbb{N}$, $\mathrm{Cu}(A_n) \simeq \mathrm{Cu}(B_n)$, that we write S_n. Now consider $\alpha_{nn+1} := \mathrm{Cu}(\phi_{nn+1})$ and $\beta_{nn+1} := \mathrm{Cu}(\psi_{nn+1})$. We have $\alpha_{nn+1},\beta_{nn+1} : S_n \longrightarrow S_{n+1}$.*
Then $dd_{\mathrm{Cu}}(\alpha_{nn+1},\beta_{nn+1}) \leq 1/3^n$.

Proof. Let $n \in \mathbb{N}$. In the same picture as in Lemma 6.4.17, we have:

$$\alpha_{nn+1} := \begin{pmatrix} \alpha^0_{nn+1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \alpha^n_{nn+1} \\ & & & \alpha^{n+1}_{nn+1} \end{pmatrix} \qquad \beta_{nn+1} := \begin{pmatrix} \beta^0_{nn+1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \beta^n_{nn+1} \\ & & & \beta^{n+1}_{nn+1} \end{pmatrix}$$

where $\alpha^i_{nn+1} := \mathrm{Cu}(\phi^i_{nn+1})$ for any $0 \leq i \leq n$ and $\alpha^{n+1}_{nn+1} := \mathrm{Cu}(\phi^{n+1}_{nn+1})$, respectively for B. We already know that $\alpha^n_{nn+1} = \beta^n_{nn+1}$ and that $\alpha^{n+1}_{nn+1} = \beta^{n+1}_{nn+1}$ since the *-homomorphisms are the same. We are now going to show that $dd_{\mathrm{Cu}}(\alpha^i_{nn+1},\beta^i_{nn+1}) \leq 1/q_i^{m_n-2}$ for any $0 \leq i \leq n$ and since $dd_{\mathrm{Cu}}(\alpha_{nn+1},\beta_{nn+1}) = \max\limits_i(dd_{\mathrm{Cu}}(\alpha^i_{nn+1},\beta^i_{nn+1}))$, the result will follow:

Let $0 \leq i \leq n-1$. Recall that $r_{n,i} := q_i^{m_n}$. Take $l_{n,i} := \max\limits_{l\in\mathbb{N}}\{l \mid 3^l < q_i^{m_n-1}\}$ and write $\{\overline{U_j}\}_{j=1}^{3^{l_{n,i}}}$ the canonical 1-thin cover of $[0,1]$ of size $1/3^{l_{n,i}}$. Observe that $1/q_i^{m_n-1} < 1/3^{l_{n,i}} \leq 1/q_i^{m_n-3}$ and hence any interval U_j of the canonical 1-thin cover contains at least q_i points of the partition $\{k/r_{n,i}\}_k$ of $[0,1]$. In fact, we write $c_j := \mathrm{card}\{k/r_{n,i} \in U_j\}$, and hence we have $c_j \geq q_i$ for any $1 \leq j \leq 3^{l_{n,i}}$. We also consider $d := \{k/r_{n,i} \in [0,1] \setminus (\bigcup\limits_{j=1}^{3^n} U_j)\}$.
Let h', h be elements of $\Gamma_{l_{n,i}}(\mathrm{Cu}(\mathcal{I}^{n-i}_{q_i}))$ such that $h' \ll h$. By construction of $\Gamma_{l_{n,i}}(\mathrm{Cu}(\mathcal{I}^{n-i}_{q_i}))$, we know that $0 \leq h', h \leq q_i.(\frac{1}{q_i^{n-i}}.1_{[0,1]})$ and hence we obtain:

$$\alpha^i_{nn+1}(h'),\beta^i_{nn+1}(h') \leq q_i.(\tfrac{q_i}{q_i^{n-i+1}}.1_{[0,1]}) + \sum_{j=1}^{3^{l_{n,i}}} c_j.h'(U_j).(\tfrac{q_i}{q_i^{n-i+1}}1_{[0,1]}) + \sum_{x\in d} h'(x).(\tfrac{q_i}{q_i^{n-i+1}}1_{[0,1]})$$
$$\alpha^i_{nn+1}(h),\beta^i_{nn+1}(h) \geq 0 + \sum_{j=1}^{3^{l_{n,i}}} c_j.h(U_j).(\tfrac{q_i}{q_i^{n-i+1}}1_{[0,1]}) + \sum_{x\in d} h(x).(\tfrac{q_i}{q_i^{n-i+1}}1_{[0,1]}).$$

where $h'(U_j), h(U_j)$ respectively denote the (constant) values of h', h on U_j.
If $h' = h$, then h is compact and hence $h = q_i.(\frac{1}{q_i^{n-i}}.1_{[0,1]})$. In this case, we easily compute that $\alpha(h) = \beta(h)$. Otherwise, we have that $\operatorname{supp} h' \ll \operatorname{supp} h$ and there exists at least one interval U_j such that $h'(U_j) < h(U_j)$. Combined with the fact that $c_j \geq q_i$, we deduce the following:

$$\sum_{j=1}^{3^{l_{n,i}}} c_j.h(U_j).(\frac{q_i}{q_i^{n-i+1}}1_{[0,1]}) \geq q_i.(\frac{q_i}{q_i^{n-i+1}}.1_{[0,1]}) + \sum_{j=1}^{3^{l_{n,i}}} c_j.h'(U_j).(\frac{q_i}{q_i^{n-i+1}}1_{[0,1]}).$$

Moreover, it is clear that $\sum_{x\in d} h'(x).(\frac{q_i}{q_i^{n-i+1}}1_{[0,1]}) \leq \sum_{x\in d} h(x).(\frac{q_i}{q_i^{n-i+1}}1_{[0,1]})$.
All of the above exactly gives us $dd_{\mathrm{Cu}}(\alpha^i_{nn+1}, \beta^i_{nn+1}) \leq 1/3^{l_{n,i}} \leq 1/q_i^{m_n-3}$ for any $0 \leq i \leq n-1$. We finally conclude that $dd_{\mathrm{Cu}}(\alpha_{nn+1}, \beta_{nn+1}) = \max_{i=0,..,n-i}(dd_{\mathrm{Cu}}(\alpha^i_{nn+1}, \beta^i_{nn+1})) \leq 1/q_0^{m_n-3} \leq 1/3^n$ by the way we chose $(m_n)_n$. □

Corollary 7.2.8. *The approximate intertwining theorem gives us* $\mathrm{Cu}(A) \simeq \mathrm{Cu}(B)$.

Proof. We only have to check the assumption (ii) of Theorem 6.4.14 as the previous lemma tells us the semimetric between morphisms of the inductive sequence goes to 0 as n tends to ∞ fast enough. Now let $n \in \mathbb{N}$, let $0 \leq i \leq n-1$ and let $l \in \mathbb{N}$. It is not hard to see that for any i, n, l in $\mathbb{N}$, we have $\alpha^i_{nn+1}(\chi_l(\mathrm{Cu}(\mathcal{I}^{n-i}_{q_i}))) \subseteq \chi_{l-1}((\mathrm{Cu}(\mathcal{I}^{n-i}_{q_i})))$ and $\beta^i_{nn+1}(\chi_l(\mathrm{Cu}(\mathcal{J}^{n-i}_{q_i}))) \subseteq \chi_{l-1}((\mathrm{Cu}(\mathcal{J}^{n-i}_{q_i})))$.
Also, for any element $s \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, $\alpha^n_{nn+1}(s), \alpha^{n+1}_{nn+1}(s), \beta^n_{nn+1}(s), \beta^{n+1}_{nn+1}(s)$ are constant lower-semicontinuous maps over $[0,1]$. A fortiori, they are l-piecewise characteristic functions for any size $l \in \mathbb{N}$ in their respective Cu-semigroups.
We can finally state that for any n, m in $\mathbb{N}$, any $m \in \mathbb{N}$, $\alpha_{nm}(\chi_l(S_n)), \beta_{nm}(\chi_l(S_n)) \subseteq \chi_l(S_m)$. We conclude using Theorem 6.4.19. □

Theorem 7.2.9. *There is no* $\mathrm{Cu}^\sim$*-isomorphism between* $\mathrm{Cu}_1(A)$ *and* $\mathrm{Cu}_1(B)$*. A fortiori,* $A \not\simeq B$.

Proof. Recall the following computations from the proof of Corollary 7.2.6:

$$\mathrm{K}_0(\mathrm{i}_n) \simeq \mathrm{K}_0(\mathrm{j}_n) \simeq \mathbb{Z}[\tfrac{1}{q_n}]$$
$$\mathrm{K}_1(\mathrm{i}_n) \simeq \mathbb{Z}/p_n\mathbb{Z} \qquad\qquad \mathrm{K}_1(\mathrm{j}_n) \simeq \mathbb{Z}/p_{n-1}\mathbb{Z}$$

Now, suppose there exists a $\mathrm{Cu}^\sim$ isomorphism $\gamma : \mathrm{Cu}_1(A) \longrightarrow \mathrm{Cu}_1(B)$. Then by everything done in Chapter 3 and more precisely in Corollary 3.3.17 and Theorem 3.3.19, we know that for any $\mathrm{Cu}_1(I) \in \mathrm{Lat}(\mathrm{Cu}_1(A))$ ideal of $\mathrm{Cu}_1(A)$ (respectively simple ideal) there exists a unique ideal $J \in \mathrm{Lat}(B)$ (respectively simple ideal) such that $\gamma(\mathrm{Cu}_1(I)) = \mathrm{Cu}_1(J)$ and such that $\gamma_{|\mathrm{Cu}_1(I)}:\mathrm{Cu}_1(I)\longrightarrow\mathrm{Cu}_1(J)$ is a $\mathrm{Cu}^\sim$-isomorphism. Moreover, the following diagram is row-exact and commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Cu}(I) & \xrightarrow{i} & \mathrm{Cu}_1(I) & \xrightarrow{j} & \mathrm{K}_1(I) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{0|\mathrm{Cu}(I)}} & & \downarrow{\scriptstyle \alpha_{|\mathrm{Cu}_1(I)}} & & \downarrow{\scriptstyle \alpha_I} & & \\ 0 & \longrightarrow & \mathrm{Cu}(J) & \xrightarrow[i]{} & \mathrm{Cu}_1(J) & \xrightarrow[j]{} & \mathrm{K}_1(J) & \longrightarrow & 0 \end{array}$$

By Lemma 7.2.5, we know that the simple ideals of $\mathrm{Cu}_1(A)$, (respectively $\mathrm{Cu}_1(B)$) are exactly $\{\mathrm{Cu}_1(\mathrm{i}_n)\}_{n\in\mathbb{N}}$, (respectively $\{\mathrm{Cu}_1(\mathrm{j}_n)\}_{n\in\mathbb{N}}$). Let $n\in\mathbb{N}$. With all the above, we know that there exists a unique $m\in\mathbb{N}$ such that $\gamma(\mathrm{Cu}_1(\mathrm{i}_n))=\mathrm{Cu}_1(\mathrm{j}_m)$ and such that $\gamma_{|\mathrm{i}_n}:\mathrm{Cu}_1(\mathrm{i}_n)\longrightarrow\mathrm{Cu}_1(\mathrm{j}_m)$ is a $\mathrm{Cu}^\sim$-isomorphism. Also by the diagram above, $\gamma_{|\mathrm{i}_n}$ induces the two following isomorphisms:

$$\begin{cases} (\gamma_{|\mathrm{i}_n})_+ : \mathrm{Cu}(\mathrm{i}_n)\simeq\mathrm{Cu}(\mathrm{j}_m) \text{ in } \mathrm{Cu} \\ (\gamma_{|\mathrm{i}_n})_{max} : \mathrm{K}_1(\mathrm{i}_n)\simeq\mathrm{K}_1(\mathrm{j}_m) \text{ in AbGp} \end{cases}$$

On the other hand, we have:

$$\begin{cases} \mathrm{K}_0(\mathrm{i}_n)\simeq\mathrm{K}_0(\mathrm{j}_m) \text{ if and only if } n=m \\ \mathrm{K}_1(\mathrm{i}_n)\simeq\mathrm{K}_1(\mathrm{j}_m) \text{ if and only if } n+1=m \end{cases}$$

We hence arrive to a contradiction since $n\neq n+1$. We conclude that $\mathrm{Cu}_1(A)\not\simeq\mathrm{Cu}_1(B)$. A fortiori, $A\not\simeq B$. □

7.2.10. (Open line of research)
We have seen that the Cu_1-semigroup captures the information of the K_1-group of any ideal of the C^*-algebra. More particularly, we have constructed an example of NCCW 1 algebras that the Cuntz semigroup could not distinguish.
It would be of interest to investigate on extending the classification results of L. Robert for NCCW 1 algebras with no K_1-obstructions by means of an augmented version of Cu (see [63]), to any NCCW 1 algebra by means of Cu_1.

Chapter 8

Classification of unitary elements of certain C^*-algebras

As already explained in Chapter 5, in the aim of classifying C^*-algebras, classification of *-homomorphisms from a subcategory of C^* into a large class of C^*-algebras has appeared to be an efficient way of finding invariants for C^*-algebras; see Theorem 5.1.9. In this setting, the Cuntz semigroup plays a major role.

In this chapter, we are going to see up to which extent the Cuntz semigroup classifies *-homomorphisms from $C(\mathbb{T})$ to A, where $A \in C^*$ is taken in the largest class possible. In the first section, we (partially) classify unitary elements of AF algebras, by means of the functor Cu together with a scale condition. In section 2 and 3, we prove that we cannot generalize this result to a larger class of C^*-algebras, by constructing examples of unitary elements first in $C([0,1]) \otimes M_{2^\infty}$ and then in the Jiang-Su algebra $\mathcal{Z}$ that agree at level of the functor Cu but fail to be approximately unitarily equivalent.

As before, we shall assume that A is a separable C^*-algebra with stable rank one. We recall that in order to ease the notations, we use C^* to denote the category of separable C^*-algebras of stable rank one. Let us first recall some correspondence between unitary elements, of a C^*-algebra A and *-homomorphisms from $C(\mathbb{T})$ to A.

8.0.1. (Unitary elements - Homomorphisms)

Let A be a unital C^*-algebra. There is a one-to-one correspondence between the set unitary elements of A, that we write $\mathcal{U}(A)$, and the set of unital *-homomorphisms from $C(\mathbb{T})$ to A, that we write $\mathrm{Hom}_{C^*,1}(C(\mathbb{T}), A)$. Let u be a unitary element of A. We define:

$$\begin{aligned}\varphi_u : C(\mathbb{T}) &\longrightarrow A\\ \mathrm{id}_{\mathbb{T}} &\longmapsto u\end{aligned}$$

Equivalently, for any $f \in C(\mathbb{T}) \supseteq C(\mathrm{sp}(u))$, we define $\varphi_u(f) := f(u)$, where $f(u)$ is obtained by functional calculus. Then, we have the following bijection:

$$\begin{aligned}\varphi : \mathcal{U}(A) &\simeq \mathrm{Hom}_{C^*,1}(C(\mathbb{T}), A)\\ u &\longmapsto \varphi_u\end{aligned}$$

We may abuse the language and say that a functor classifies unitary elements of A whenever it classifies homomorphisms from $C(\mathbb{T})$ to A; see Definition 5.1.5.

Definition 8.0.2. Let A be a unital C^*-algebra. Let u, v be unitary elements in A. We say that u and v are *approximately unitarily equivalent*, and we write $u \sim_{aue} v$, if $\inf\limits_{w \in \mathcal{U}(A)} \|wuw^* - v\| = 0$.

8.0.3. Let us define two pseudometrics on $\mathrm{Hom}_1(C(\mathbb{T}), A)$. Recall that we denote the open sets of $\mathbb{T}$ by $O(\mathbb{T})$.

Definition 8.0.4. [41, Definition 3.1] Let A be a unital C^*-algebra. For any $U \in O(\mathbb{T})$, and any $r > 0$, we define an *r-open neighborhood of U*, that we write $U_r := \bigcup\limits_{x \in U} B(x, r)$; see Definition 6.2.7.
Now, for any two *-homomorphisms $\varphi_u, \varphi_v : C(\mathbb{T}) \longrightarrow A$, we define:

$$d_{\mathrm{Cu}}(\varphi_u, \varphi_v) := \inf\{r > 0 \mid \forall U \in O(X), \varphi_u(f_U) \precsim_{\mathrm{Cu}} \varphi_v(f_{U_r}) \text{ and } \varphi_v(f_U) \precsim_{\mathrm{Cu}} \varphi_v(f_{U_r})\}$$

where U_r is an r-open neighborhood of U and $f_U : \mathbb{T} \longrightarrow \mathbb{R}_+$ is a continuous function such that $\mathrm{supp}(f_U) = U$. We refer to d_{Cu} as the *Cuntz pseudometric*. We also define:

$$d_U(\varphi_u, \varphi_v) := \inf\{\epsilon > 0 \mid \forall F \underset{finite}{\subseteq} C(\mathbb{T})_1, \exists w \in \mathcal{U}(A) : \|w\varphi_u(f)w^* - \varphi_v(f)\| < \epsilon, \forall f \in F\}.$$

We refer to d_U as the *unitary pseudometric*.
For both constructions, if the infimum defined does not exist, we set the value to ∞.

Remark 8.0.5. For a C^* algebra A, we can define a unitary pseudometric over $\mathcal{U}(A)$ as the distance between unitary orbits of two unitary elements of A as follows: Let u, v be unitary elements in A, then $d_U(u, v) := \inf\limits_{w \in \mathcal{U}(A)} \|wuw^* - v\|$.

Proposition 8.0.6. *Let A be a unital C^*-algebra. Let u, v be unitary elements of A. Then:*
(i) $d_U(u, v) \leq d_U(\varphi_u, \varphi_v)$.
(ii) $d_{\mathrm{Cu}}(\varphi_u, \varphi_v) \leq d_U(u, v)$.
(iii) $d_{\mathrm{Cu}}(\varphi_u, \varphi_v) = d_{\mathrm{Cu}}(\mathrm{Cu}(\varphi_u), \mathrm{Cu}(\varphi_v))$, see Definition 6.2.7.
Also the following conditions are equivalent:
(iv) $d_U(u, v) = 0$ if and only if $d_U(\varphi_u, \varphi_v) = 0$ if and only if $u \sim_{aue} v$ if and only if $\varphi_u \sim_{aue} \varphi_v$.

Proof. One can use functional calculus to obtain the proposition. □

8.1 Classification of unitary elements of AF algebras

8.1.1. In this section, we classify homomorphisms from $C(\mathbb{T})$ to any AF algebra by means of the functor Cu. We mention that so far, only the uniqueness condition has been proved; see Definition 5.1.5. To do so, we first prove an abstract version of the theorem in the category Cu. We will hence use notations and tools from Chapter 6 and we refer the reader there for some definitions/properties.

Theorem 8.1.2. *Let X be the circle or the interval and let $n \in \mathbb{N}$. Let $(S_i, \sigma_{ij})_{i\in\mathbb{N}}$ be an inductive sequence in the category* Cu *and $(S, \sigma_{i\infty})_{i\in\mathbb{N}}$ its direct limit.*
Let $\alpha, \beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow S$ be Cu*-morphisms that factorize through a S_i for some $i \in \mathbb{N}$, that is, there exist $\alpha_i, \beta_i : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow S_i$ such that $\alpha = \sigma_{i\infty} \circ \alpha_i$ and $\beta = \sigma_{i\infty} \circ \beta$. If $\alpha \underset{\Gamma_n}{\approx} \beta$, then there exists some $j \geq i$ such that $\sigma_{ij} \circ \alpha_i \underset{\Gamma_{n-1}}{\approx} \sigma_{ij} \circ \beta_i$.*

Proof. Let $\alpha, \beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow S$ be Cu-morphisms that factorize through S_i for some $i \in \mathbb{N}$, such that $\alpha \underset{\Gamma_n}{\approx} \beta$. For any $g, g'' \in \Gamma_{n-1}$ such that $g \ll g''$, we can find some $g' \in \Gamma_n$ such that $g \ll g' \ll g''$ in Γ_n (see Lemma 6.1.15). Furthermore, using the hypothesis, we know that $\sigma_{i\infty} \circ \alpha_i(g') \ll \sigma_{i\infty} \circ \beta_i(g'')$ and $\sigma_{i\infty} \circ \beta_i(g') \ll \sigma_{i\infty} \circ \alpha_i(g'')$. Thus, using (L2) of Proposition 6.3.2, we deduce that there exists some $j \geq i$ such that $\sigma_{ij} \circ \alpha_i(g) \ll \sigma_{ij} \circ \beta_i(g'')$ and $\sigma_{ij} \circ \beta_i(g) \ll \sigma_{ij} \circ \alpha_i(g'')$. Finally, since Γ_n is a finite set, we can find some $j \in \mathbb{N}$ big enough such that $\sigma_{ij} \circ \alpha_i \underset{\Gamma_{n-1}}{\approx} \sigma_{ij} \circ \beta_i$.

□

8.1.3. (Open line of research)
The original version of Theorem 8.1.2 was also including the following:
Let $\alpha : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow S$ be a Cu-morphism. Then there exists $i \in \mathbb{N}$ and a Cu-partial

morphism $\alpha_i : \Gamma_n \subseteq \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow S_i$ such that $\alpha \underset{\Gamma_n}{\approx} \sigma_{i\infty} \circ \alpha_i$.
However, the proof is still on-going. Note that this would allow us to obtain the existence part of the classification of unitary elements of AF algebras by means of the functor Cu.

8.1.4. We will now apply this abstract theorem to homomorphisms between $\mathrm{Cu}(C(\mathbb{T}))$ and $\mathrm{Cu}(A)$, where A is any AF algebra. For the rest of this section, we denote $\{\overline{U_k}\}_{k=1}^{3^n}$ the canonical 1-thin cover of $\mathbb{T}$ of size $1/3^n$ and as in Chapter 6, we denote Γ_n to be the set of n-piecewise characteristic functions of $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ taking values in $\{0, 1\}$. See Paragraph 6.1.9.

Definition 8.1.5. A *bipartite graph* is a graph whose vertices can be divided in two disjoint sets U, V such that every edge connects a vertex of U to one of V. We often write $G = (U + V, E)$.

Definition 8.1.6. Let $G = (E^0, E^1)$ be a graph. A *matching* is a subset $F \subseteq E^1$ such that no two elements of F share an endpoint. That is, all elements of E^0 are endpoints of at most one element of F.
Let $G := (X + Y, E)$ be a finite bipartite graph. By an *X-saturating matching*, we refer to any matching that covers every vertex in X.

Theorem 8.1.7. *(Hall's marriage theorem)*
Let $G := (X + Y, E)$ be a finite bipartite graph with bipartite sets X and Y. Let $W \subseteq X$. We define $n_G(W) := \bigcup_{w \in W} \{y \in Y : (w, y) \in E\}$, that is, $n_G(W)$ consists of all the vertices in Y that are linked with some w in W. Then the following are equivalent:
(i) There exists an X-saturating matching.
(ii) For any $W \subseteq X$, $\#W \leq \#n_G(W)$.

Theorem 8.1.8. *Let B be any finite dimensional C^*-algebra and let $n \in \mathbb{N}$.*
(i) Let u, v be unitary elements of B such that $\mathrm{Cu}(\varphi_u) \underset{\Gamma_n}{\approx} \mathrm{Cu}(\varphi_v)$. Then, there exists a unitary element $w \in B$ such that $\|wuw^* - v\| < 1/3^n$.
(ii) Let $\alpha : \Gamma_n \subset \mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}) \longrightarrow \mathrm{Cu}(B)$ be a Cu-partial morphism such that $\alpha(1_{|\mathbb{T}}) = [1_B]$. Then there exists a unitary $u_\alpha \in B$ such that $\mathrm{Cu}(\varphi_{u_\alpha}) \underset{\Gamma_n}{\approx} \alpha$.

Proof. Let B be any finite dimensional C^*-algebra and let $n \in \mathbb{N}$. We first suppose that $B \simeq M_l(\mathbb{C})$ for some $l \in \mathbb{N}$ and the general case will follow as a consequence. We denote the canonical 1-thin cover of the circle by $\{U_k\}_{k=1}^{3^n}$.
(i) Let u, v be unitary elements of B such that $\mathrm{Cu}(\varphi_u) \underset{\Gamma_n}{\approx} \mathrm{Cu}(\varphi_v)$ in $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$. Let $X := \mathrm{sp}(u) \subseteq \mathbb{T}$, $Y := \mathrm{sp}(v) \subseteq \mathbb{T}$ and $E := \{(x, y) \in X \times Y : \|x - y\| < 1/3^n\})$. Let W be a finite subset of X.

Arguing similarly as in the proof of Proposition 6.2.10), we know that there exists a minimal element γ of Γ_n such that $W \subseteq \operatorname{supp}\gamma$. We write $V := \operatorname{supp}\gamma$. Observe that V has a finite number of (open) connected components. First, suppose that V is an open connected set.
Now, let f_γ be a continuous function over the circle with values in $\mathbb{R}_+$ such that $\operatorname{supp}(f_\gamma) = V$. That is, $[f_\gamma] = \gamma$. Hence, there exist $1 \leq r \leq s \leq 3^n$ such that $V := U_r \cup (\bigcup_{k=r+1}^{s-1} \overline{U_k}) \cup U_s$. We consider $V' := U_{r-1} \cup (\bigcup_{k=r}^{s} \overline{U_k}) \cup U_{s+1}$ and we write $\gamma' := 1_{V'} \in \Gamma_n$. Obviously, $\gamma \ll \gamma'$ in Γ_n. In fact, it is easy to see that $\gamma' = \min\{h \in \Gamma_n \mid \gamma \ll h\}$. Again, let $f_{\gamma'}$ be a continuous function over the circle with values in $\mathbb{R}_+$ such that $\operatorname{supp}(f_{\gamma'}) = V'$.
We know from hypothesis that $[\varphi_u(f_\gamma)] \ll [\varphi_v(f_{\gamma'})]$ in $\overline{\mathbb{N}}$. Also, notice that $\operatorname{card}(W) = [\varphi_u(f_\gamma)]$ and $[\varphi_v(f_{\gamma'})] \leq \operatorname{card}(n_G(W))$, -indeed $\sup_{y\in V'} d(y,V) = 1/3^n$-. Thus we get that $\operatorname{card}(W) \leq \operatorname{card}(n_G(W))$ and hence condition (ii) of Theorem 8.1.7 holds for the finite bipartite graph $(X+Y,E)$. So there exists an X-saturating matching in E.
Since $\mathbb{T}$ is compact, we obtain that $\operatorname{card}(X) = \operatorname{card}(Y)$ and hence any X-saturating matching is in fact a perfect matching. Thus, there exists a bijection σ between X and Y such that $\|\sigma\| < 3/3^n$. In other words, $d_U(u,v) < 1/3^n$ and hence we conclude that there exists a unitary $w \in B$ such that $\|wuw^* - v\| < 1/3^n$. Finally, as mentioned above, V has a finite number of (open) connected components so it is easy to check that (i) follows.

(ii) Let α be as in (ii). Write $x_{k-1} := e^{2ik\pi/3^n}$ and $y_k := e^{2i\pi/(2.3^n)}x_k$ for any $1 \leq k \leq 3^n$. Observe that $U_k =]x_{k-1}; x_k[$ and that y_k is the center of U_k, for any $1 \leq k \leq 3^n$ (convention: $x_{3^n} = x_0$). We will recursively build a unitary element of B that has the required properties:
Let u be an empty matrix. For every $1 \leq k \leq 3^n - 1$, set $W_k := U_k \cup U_{k+1} \cup \{x_k\}$ and consider $p := \alpha([f_{U_k}]), q := \alpha([f_{U_{k+1}}]), r := \alpha([f_{W_k}])$. Now, apply the following steps for $1 \leq k \leq 3^n$:
(1) $u = u \oplus \operatorname{diag}_q(y_k)$.
(2) If $p + q < r$, then $u = u \oplus \operatorname{diag}_{r-(p+q)}(x_k)$. (Convention, $U_{3^n+1} = U_1$.)
We denote the element obtained by u_α. By construction, u_α is a finite unitary matrix. Let v be the minimal element of Γ_n that contains $\operatorname{sp}(u_\alpha)$ and consider $\beta := \operatorname{Cu}(\varphi_{u_\alpha})$. We already know that $\alpha([f_{U_k}]) = \beta([f_{U_k}])$, for any k. Now consider $V := U_j \cup (\bigcup_{k=j+1}^{l-1} \overline{U_k}) \cup U_l$. Observe that $V \bigsqcup (\bigcup_{k=j+1}^{l-1} U_k) = \bigsqcup_{k=j}^{l-2} W_k$. Hence using (iii) of Definition 6.2.2, we get that $\alpha(1_V) + \sum_{j+1}^{l-1} \alpha(1_{U_k}) = \sum_{j}^{l-2} \alpha(1_{W_k})$. The same holds for β.
Furthermore, by construction, we have that $\alpha(1_{W_k}) \leq \beta(1_{W_k})$, for any $1 \leq k \leq 3^n$. Note that

$\alpha(1_{W_k}) = \beta(1_{W_k})$ whenever $W_k \subseteq \operatorname{supp} \nu$. Putting everything together, we obtain

$$\alpha(1_V) + \sum_{j+1}^{l-1} \alpha(1_{U_k}) \leq \beta(1_V) + \sum_{j+1}^{l-1} \beta(1_{U_k}).$$

Since $\overline{\mathbb{N}}$ has cancellation, we deduce that $\sum\limits_{j+1}^{l-1} \alpha(1_{U_k}) = \sum\limits_{j+1}^{l-1} \beta(1_{U_k})$ and we conclude that $\alpha(1_V) \leq \beta(1_V) \ll \beta(1_{V'})$, for any $V \ll V'$.
On the other hand, combining the fact that $\alpha(1_{W_k}) = \beta(1_{W_k})$ whenever $1_{W_k} \subseteq \operatorname{supp} \nu$ with the argument above, we deduce that $\beta(1_V) = \alpha(1_V)$ whenever $V \subseteq \operatorname{supp} \nu$. In fact, observe that $\beta(1_V) = \beta(1_{V \cap \operatorname{supp} \nu})$ and finally compute $\beta(1_{V \cap \operatorname{supp} \nu}) = \alpha(1_{V \cap \operatorname{supp} \nu}) \ll \alpha(1_{V'})$, for any $V \ll V'$, from which we conclude that $\alpha \underset{\Gamma_n}{\approx} \beta$.
Finally, we know that $1_{\mathbb{T}}$ is a compact element of $\operatorname{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ and hence $\beta(1_{\mathbb{T}}) \ll \alpha(1_{\mathbb{T}})$ and $\beta(1_{\mathbb{T}}) \gg \alpha(1_{\mathbb{T}})$. Since $\ll$ and $\leq$ agree on $\overline{\mathbb{N}}$, we obtain $\beta(1_{\mathbb{T}}) = \alpha(1_{\mathbb{T}})$. This scale condition $\alpha(1_{\mathbb{T}}) = l$ gives us that $u_\alpha \in M_l(\mathbb{C})$, which ends the proof. □

Remark 8.1.9. Observe that for two unitary elements u, v of a finite dimensional C^*-algebra B, we have the following implications: $d_U(u, v) \leq 1/3^n \implies d_{\mathrm{Cu}}(\varphi_u, \varphi_v) \leq 1/3^n \implies \mathrm{Cu}(\varphi_u) \underset{\Gamma_{n-1}}{\approx} \mathrm{Cu}(\varphi_v) \implies d_U(u, v) < 1/3^{n-1}$. Thus, we conjecture that $d_U(u, v) = d_{\mathrm{Cu}}(\varphi_u, \varphi_v)$.

Corollary 8.1.10. *Let* $A := \lim\limits_{\to n}(A_n, \phi_{nm})$ *be a (unital)* AF *algebra. Let* u, v *be unitary elements of* A *such that* $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$. *Then* $u \sim_{aue} v$.

Proof. Let u, v be unitary elements of A such that $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$. One can check that we can find two unitary elements u_m, v_m of A_m for some $m \in \mathbb{N}$, such that $\|\phi_{m\infty}(u_m) - u\| < 1/(2.3^n)$ and $\|\phi_{m\infty}(v_m) - v\| < 1/(2.3^n)$.
Since $d_{\mathrm{Cu}}(\phi_{m\infty} \circ \varphi_{u_m}, \phi_{m\infty} \circ \varphi_{v_m}) \leq d_{\mathrm{Cu}}(\phi_{m\infty} \circ \varphi_{u_m}, \varphi_u) + d_{\mathrm{Cu}}(\varphi_u, \varphi_v) + d_{\mathrm{Cu}}(\varphi_v, \phi_{m\infty} \circ \varphi_{v_m})$, we deduce that $d_{\mathrm{Cu}}(\phi_{m\infty} \circ \varphi_{u_m}, \phi_{m\infty} \circ \varphi_{v_m}) < 2/(2.3^n) = 1/3^n$.
By Proposition 6.2.10, we know that $\mathrm{Cu}(\phi_{m\infty} \circ \varphi_{u_m}) \underset{\Gamma_n}{\simeq} \mathrm{Cu}(\phi_{m\infty} \circ \varphi_{v_m})$ and using Lemma 6.2.9, we get that $\mathrm{Cu}(\phi_{m\infty} \circ \varphi_{u_m}) \underset{\Gamma_{n-1}}{\approx} \mathrm{Cu}(\phi_{m\infty} \circ \varphi_{v_m})$. Now we apply Theorem 8.1.2: there exists some $p \in \mathbb{N}$ such that $\mathrm{Cu}(\phi_{mp} \circ \varphi_{u_m}) \underset{\Gamma_{n-2}}{\approx} \mathrm{Cu}(\phi_{mp} \circ \varphi_{v_m})$. As stated in Remark 8.1.9, this finally gives us that $d_U(\phi_{mp}(u_m), \phi_{mp}(v_m)) < 1/3^{n-2}$. Thus, there exists a unitary $w \in A_p$ such that

$\|Ad(w) \circ \phi_{mp}(u_m) - \phi_{mp}(v_m)\| < 1/3^{n-2}$. Now we obtain:

$$\begin{aligned}\|Ad(\phi_{p\infty}(w)) \circ u - v\| &\leq \|Ad(\phi_{p\infty}(w)) \circ u - Ad(\phi_{p\infty}(w)) \circ \phi_{p\infty}(\phi_{mp}(u_m))\| \\ &\quad + \|Ad(\phi_{p\infty}(w)) \circ \phi_{p\infty}(\phi_{mp}(u_m)) - \phi_{p\infty}(\phi_{mp}(v_m))\| \\ &\quad + \|\phi_{p\infty}(\phi_{mp}(v_m)) - v\| \\ &< 1/3^n + 1/3^{n-2} \qquad\qquad (\phi_{p\infty} \text{ is a contraction}) \\ \|Ad(\phi_{p\infty}(w)) \circ u - v\| &< 2/3^{n-2}.\end{aligned}$$

We conclude that $\varphi_u \sim_{aue} \varphi_v$. Equivalently, $u \sim_{aue} v$. □

8.2 An example in $C[0,1] \otimes M_{2^\infty}$

8.2.1. In this section, we exhibit an example in a rather simple setting. We will construct two unitary elements of $C[0,1] \otimes M_{2^\infty}$ such that $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$ but u and v are not approximately unitarily equivalent. This shows that more information is needed in order to extend the classification results of Section 8.1 to a larger class than AF algebras.

8.2.2. Let us first make precise the setting we will be working in. We refer the reader to Chapter 4 for definitions and properties regarding UHF algebras. Consider the supernatural number of infinite type 2^∞ and its associated UHF algebra $M_{2^\infty} := \varinjlim_n (\bigotimes_{k=0}^{n} M_2(\mathbb{C}), \psi_{nm})$, where $\psi_{nm} := id \otimes 1_{2^{m-n}}$. Write $\phi_{nm} := \mathrm{id}_{C[0,1]} \otimes \psi_{nm}$. It is trivial to see that $\varinjlim_n (C[0,1] \otimes (\bigotimes_{k=0}^{n} M_2(\mathbb{C})), \phi_{nm}) \simeq C[0,1] \otimes M_{2^\infty}$. We will now construct recursively unitary elements in $C[0,1] \otimes M_{2^\infty}$ that will be our counter example.

Set as a convention $\bigotimes_{k=0}^{0} M_2(\mathbb{C}) \simeq \mathbb{C}$. Define $A := (C[0,1] \otimes M_{2^\infty}, \phi_{n\infty})$. For any $n \in \mathbb{N}$, consider $h_n := \mathrm{diag}(k/2^n)_{k=0}^{2^n-1}$. It is clear that h_n is a positive element of $\bigotimes_{k=0}^{n} M_2(\mathbb{C})$. Now define $w_n := e^{2i\pi h_n}$. It is also clear that w_n is a unitary element of $\bigotimes_{k=0}^{n} M_2(\mathbb{C})$. Observe that for any $n \in \mathbb{N}$, there exists a unitary permutation p_n of $\bigotimes_{k=0}^{n+1} M_2(\mathbb{C})$ such that $\|p_n \psi_{nn+1}(h_n) p_n^* - h_{n+1}\| \leq 1/2^n$. Define $h'_0 := h_0$ and $h'_n := p_{n-1} h_n p_{n-1}^*$ for any $n > 0$.

We now compute that $\|\psi_{n+1n}(h'_n) - h'_{n+1}\| \leq 1/2^n$ and hence, $(\psi_{n\infty}(h'_n))_n$ is a Cauchy sequence. So it converges towards a positive element $h \in M_{2^\infty}$. Now write $w'_n := e^{2i\pi h'_n}$. Using functional calculus, we obtain that $(\psi_{n\infty}(w'_n))_n$ converges towards $w := e^{2i\pi h}$. It is clear that w a unitary element $w \in M_{2^\infty}$ and observe that $\mathrm{sp}(w) = \mathbb{T}$.

Now write $g_1 := 1_{|[0,1]}$ and $g_2 := \mathrm{id}_{[0,1]}$. Define $u := e^{2i\pi g_1} \otimes w$ and $v := e^{2i\pi g_2} \otimes w$. Also define $u_n := e^{2i\pi g_1} \otimes w'_n$ and $v_n := e^{2i\pi g_2} \otimes w'_n$.
Notice that we can think of u and v as unitary elements of $C([0,1], M_{2^\infty})$ (respectively u_n, v_n in $C([0,1], \overset{n}{\underset{k=0}{\otimes}} M_2(\mathbb{C}))$). Indeed:

$$u : [0,1] \longrightarrow M_{2^\infty} \qquad\qquad v : [0,1] \longrightarrow M_{2^\infty}$$
$$t \longmapsto w \qquad\qquad t \longmapsto e^{2i\pi t}.w$$

Besides, since $e^{2i\pi t}.e^{2i\pi h} = e^{2i\pi w}.e^{2i\pi t}$ for every $t \in [0,1]$, we can even rewrite $v(t) = e^{2i\pi(h+t)}$. We now need to check that u, v as constructed are unitary elements of A with the required properties. We claim the following:

Claim 1: $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$.
Claim 2: u is not approximately unitarily equivalent to v.

Proof of Claim 1: In the first place, we will prove that $d_{\mathrm{Cu}}(\varphi_{u_n}, \varphi_{v_n}) \leq 1/2^{n+1}$. Observe that u_n, v_n are unitary elements of $C([0,1], \overset{n}{\underset{k=0}{\otimes}} M_2(\mathbb{C}))$ and that $\mathrm{Cu}(\varphi_{u_n}), \mathrm{Cu}(\varphi_{v_n}) : \mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}) \longrightarrow \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$. Thus, $\mathrm{Cu}(\varphi_{u_n})(1_{|U}), \mathrm{Cu}(\varphi_{v_n})(1_{|U}) \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ for any $U \in \mathcal{O}(\mathbb{T})$. Since the order is pointwise in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, we obtain $d_{\mathrm{Cu}}(\varphi_{u_n}, \varphi_{v_n}) = \sup\limits_{t\in[0,1]} d_{\mathrm{Cu}}(\varphi_{u_n(t)}, \varphi_{v_n(t)})$.
On the other hand, $u_n(t), v_n(t)$ are diagonal unitary matrices in $\overset{n}{\underset{k=0}{\otimes}} M_2(\mathbb{C})$, we can picture them as 2^n points of $\mathbb{T}$ -consisting of the elements of the diagonal-. In fact, for any $U \in \mathcal{O}(\mathbb{T})$, the natural numbers $\mathrm{Cu}(\varphi_{u_n})(1_{|U})(t), \mathrm{Cu}(\varphi_{v_n})(1_{|U})(t)$ correspond to the number of diagonal entries of $u_n(t), v_n(t)$ that belong to U.
Thus, we obtain that $d_{\mathrm{Cu}}(\varphi_{u_n(t)}, \varphi_{v_n(t)}) := \max\limits_{i,j}\{\, \|(u_n(t))_{ii} - (v_n(t))_{jj}\|_{\mathbb{T}}\,\}$. By construction, we compute that $\sup\limits_{t\in[0,1]} d_{\mathrm{Cu}}(\varphi_{u_n(t)}, \varphi_{v_n(t)}) = 1/2^{n+1}$ and we deduce that $d_{\mathrm{Cu}}(\varphi_{u_n}, \varphi_{v_n}) \leq 1/2^{n+1}$.
Let $\epsilon > 0$. We know that there exists $n \in \mathbb{N}$ such that $\|\psi_{n\infty}(w'_n) - w\| < \epsilon/2$, and hence $\|u - e^{2i\pi g_1} \otimes \psi_{n\infty}(w'_n)\| < \epsilon/2$, respectively v, g_2. Now, we have:

$$\begin{aligned} d_{\mathrm{Cu}}(\varphi_u, \varphi_v) &\leq d_{\mathrm{Cu}}(\varphi_u, \varphi_{e^{2i\pi g_1}\otimes\psi_{n\infty}(w'_n)}) + d_{\mathrm{Cu}}(\varphi_{e^{2i\pi g_1}\otimes\psi_{n\infty}(w'_n)}, \varphi_{e^{2i\pi g_2}\otimes\psi_{n\infty}(w'_n)}) + d_{\mathrm{Cu}}(\varphi_{e^{2i\pi g_1}\otimes\psi_{n\infty}(w'_n)}, \varphi_v) \\ &\leq \epsilon/2 + d_{\mathrm{Cu}}(\varphi_{u_n}, \varphi_{v_n}) + \epsilon/2 \qquad (\phi_{n\infty} \text{ is a unital isometry}) \\ d_{\mathrm{Cu}}(\varphi_u, \varphi_v) &\leq \epsilon + 1/2^{n+1}. \end{aligned}$$

Claim 1 follows by making n tend to ∞.

Before proving *Claim 2*, we will first recall the definition of a determinant defined for C^*-algebras in [38] and some more properties that can be found in [51] and [74].
We also recall that for a (multiplicative) group G, a *commutator* is an element of the form $[g,h] := ghg^{-1}h^{-1}$. We usually write DG the normal subgroup of G generated by the commutators of G.

Lemma 8.2.3. *Let A be a separable unital C^*-algebra of stable rank one. Then:*
(i) $D\mathcal{U}(A) \subseteq \mathcal{U}^0(A)$.
(ii) $u \sim_{aue} v$ *implies that* $u \sim_h v$.
(iii) $0 \longrightarrow \mathcal{U}^0(A)/D\mathcal{U}(A) \xrightarrow{i} \mathcal{U}(A)/D\mathcal{U}(A) \xrightarrow{\pi} \mathrm{K}_1(A) \longrightarrow 0$ *is a split-exact sequence in AbGp.*

Proof. All of these are to be found in [74, Section 3]. We still give a proof, for the sake of completeness. (i) Observe that

$$\begin{pmatrix} uvu^{-1}v^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}\begin{pmatrix} u^{-1} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v \end{pmatrix}$$

From which we deduce $D\mathcal{U}(A) \subseteq D\mathcal{U}^0(M_3(A))$. On the other hand, since A has stable rank one, we know that $\mathcal{U}(A)/\mathcal{U}_0(A) \simeq \mathrm{K}_1(A)$ (see Proposition 1.1.17). Thus we can find $u_0, v_0 \in \mathcal{U}_0(A)$ such that $\pi(u_0) = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \pi(v_0) = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}$ which ends the proof.
(ii) Let $u \sim_{aue} v$. We can find a unitary w such that $\|u - wvw^*\| < 2$. We get that $u \sim_h wvw^*$. Going to matrices, we have in $M_2(\mathbb{C})$: $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} wvw^* & 0 \\ 0 & 1 \end{smallmatrix}\right) \sim_h \left(\begin{smallmatrix} v & 0 \\ 0 & ww^* \end{smallmatrix}\right)$. Finally using that $\mathcal{U}(A)/\mathcal{U}_0(A) \simeq \mathrm{K}_1(A)$ again, we conclude that $u \sim_h v$ in $\mathcal{U}(A)$.
(iii) Injectivity of i is trivial. Using (i), surjectivity of π becomes clear as well. We also have $\mathrm{im}(i) \subseteq \ker(\pi)$. We now have to check that $\ker(\pi) \subseteq \mathrm{im}(i)$. Let $[u] \in \ker(\pi)$ and let u be a representative of $[u]$. Since $[u] \in \ker(\pi)$, we have that $u \sim_h 1$, that is, $[u] \in \mathcal{U}^0(A)/D\mathcal{U}(A) = \mathrm{im}(i)$. □

Definition 8.2.4. [38, Section 2]
Let A be a (unital) C^*-algebra. We define the *universal trace* on A as the quotient morphism $Tr : A \longrightarrow A_q := A/\overline{[A,A]}$. We extend this map to $M_\infty(A)$ and we define

$$\begin{aligned} \underline{Tr} : \mathrm{K}_0(A) &\longrightarrow A_q \\ [e] &\longmapsto Tr(e) \end{aligned}$$

We finally call the *de La Harpe-Skandalis determinant associated to Tr*, the group homomorphism $\Delta_{Tr} : Gl^0_\infty(A) \longrightarrow A_q/\underline{Tr}(\mathrm{K}_0(A))$ such that $\Delta_{Tr}(e^{2i\pi y}) = [Tr(y)]$ for all $y \in M_\infty(A)$. Moreover, $\overline{DGl^0_\infty(A)} \subseteq \ker(\Delta_{Tr})$.

Corollary 8.2.5. *Let u, v be elements of $\mathcal{U}^0(A)$ such that $u \sim_{aue} v$. Then $\Delta_{Tr}(u) = \Delta_{Tr}(v)$.*

Proof. Let us first show that Δ_{Tr} is constant on $\sim_{ue}$-classes of $\mathcal{U}^0(A)$. Let $u \in \mathcal{U}^0(A)$ and $w \in \mathcal{U}(A)$. Since $wuw^* \in \mathcal{U}^0(A)$, we easily see that $wuw^*u^* = wuw^{-1}u^{-1} \in D\mathcal{U}(A) \cap \mathcal{U}^0(A)$. We also know that $\overline{D\mathcal{U}(A) \cap \mathcal{U}^0(A)} \subseteq \overline{D\mathcal{U}^0(M_3(A))} \subseteq \overline{DGl^0_\infty(A)} \subseteq \ker(\Delta_{Tr})$. We deduce that $\Delta_{Tr}(wuw^*u^*) = 0$. That is, $\Delta_{Tr}(wuw^*) = \Delta_{Tr}(u)$. Arguing similarly, we extend our result to $\sim_{aue}$-classes of $\mathcal{U}^0(A)$. □

8.2.6. *Proof of Claim 2*: Let us compute $\Delta_{Tr}(u)$ and $\Delta_{Tr}(v)$. Notice that M_{2^∞} has a unique trace, that happens to be the universal trace. That is, $(M_{2^\infty})_q \simeq \mathbb{C}$. We can then describe $A_q \simeq C[0,1]$ by sending $f \in A$ to $(t \longmapsto [f(t)]_{(M_{2^\infty})_q}) \in C[0,1]$. Moreover, we have $\mathrm{K}_0(M_{2^\infty}) \simeq \mathbb{Z}[\frac{1}{2}]$ and hence, $\mathrm{K}_0(A) \simeq \{k.1_{|[0,1]}, k \in \mathrm{K}_0(M_{2^\infty})\} \simeq \mathbb{Z}[\frac{1}{2}]$.
Putting it all together, we get that $\underline{Tr}(\mathrm{K}_0(A)) \simeq \{k.1_{|[0,1]}, k \in \mathbb{Z}[\frac{1}{2}]\}$. So we finally have $A_q/\underline{Tr}(\mathrm{K}_0(A)) \simeq C[0,1]/(\{k.1_{|[0,1]}, k \in \mathbb{Z}[\frac{1}{2}]\}) \simeq C([0,1], \mathbb{C}/\mathbb{Z}[\frac{1}{2}])$.
Now we compute $\Delta_{Tr}(u) = (t \longmapsto [Tr(h_w)]_{\mathbb{C}/\mathbb{Z}[\frac{1}{2}]})$ and $\Delta_{Tr}(v) = (t \longmapsto [Tr(h_w) + t]_{\mathbb{C}/\mathbb{Z}[\frac{1}{2}]})$. They are clearly distinct. The result follows using the contraposition of Corollary 8.2.5.

8.3 An example in the Jiang-Su algebra

8.3.1. In this third section, we exhibit two unitary elements of the Jiang-Su algebra $\mathcal{Z}$ that agree at level of Cu but fail to be approximately unitarily equivalent. We first give a definition and usual properties of $\mathcal{Z}$. We refer the reader to [68], [43] for more details on the Jiang-Su algebra and to Paragraph 4.4.1 for the construction of NCCW 1.

8.3.2. *Dimension-drop interval algebras*:
A generalization of Elliott-Thomsen algebras gave rise to what is commonly known as *dimension drop interval* algebras: Let p, q be natural numbers. We define $Z_{pq} := A(M_p \oplus M_q, M_p \otimes M_q, \pi_0 \otimes 1_q, 1_p \otimes \pi_1)$, where $\pi_0 : M_p \oplus M_q \longrightarrow M_p$ and $\pi_1 : M_p \oplus M_q \longrightarrow M_q$ are the respective projections on each component of the direct sum. In case p and q are coprime, Z_{pq} is called a *prime dimension drop* algebra.

Write d :=gcd(p, q). By Proposition 4.4.4, we know that:

$$\mathrm{K}_0(Z_{pq}) \simeq \mathbb{Z} \qquad \mathrm{Cu}(I_q) \simeq \{f \in \mathrm{Lsc}([0,1], \overline{\mathbb{N}}) \mid f(0) \in q\overline{\mathbb{N}}, f(1) \in p\overline{\mathbb{N}}\}$$
$$\mathrm{K}_1(Z_{pq}) \simeq \mathbb{Z}/d\mathbb{Z} \qquad \simeq \{f \in \mathrm{Lsc}([0,1], \tfrac{1}{pq}\overline{\mathbb{N}}) \mid f(0) \in \tfrac{1}{p}\overline{\mathbb{N}}, f(1) \in \tfrac{1}{q}\overline{\mathbb{N}}\}$$

The Jiang-Su algebra:
The Jiang-Su algebra, commonly denoted by $\mathcal{Z}$, is an infinite-dimensional unital (separable) strongly self-absorbing simple C^*-algebra of stable rank one. $\mathcal{Z}$ has a unique tracial state and the same K-Theory as $\mathbb{C}$. It was first constructed in [43].

A way to construct $\mathcal{Z}$ is using results of [68, Theorem 3.4]: Let p, q be infinite supernatural numbers that are coprime. Then there exists a trace-collapsing unital endomorphism on $Z_{pq} := \varinjlim_n (Z_{p^n q^n}, i_n)$. That is, a unital endomorphism such that $\tau \circ \varphi = \tau' \circ \varphi$ for any pair of tracial states τ, τ' on A.
Then $\mathcal{Z} \simeq \varinjlim (Z_{pq}, \varphi)$, where φ is any trace-collapsing unital endomorphism on Z_{pq}. For instance, one can write $\mathcal{Z} \simeq \varinjlim (Z_{2^\infty 3^\infty}, \varphi)$. Finally, by [43, Theorem 1] and by [3, Section 4], or else [4, §7.6.1] and Proposition 4.1.1, we have:

$$\mathrm{K}_0(\mathcal{Z}) \simeq \mathbb{Z} \qquad \mathrm{Cu}(\mathcal{Z}) \simeq \mathbb{N} \sqcup]0, \infty]$$
$$\mathrm{K}_1(\mathcal{Z}) \simeq 0 \qquad \mathrm{Cu}_1(\mathcal{Z}) \simeq (\mathbb{N} \sqcup]0, \infty]) \times \{0\}$$

Theorem 8.3.3. *(Riesz-Markov - see [71, §6.3] for more details on Radon measures)*
Let X be a Hausdorff and locally compact space. Let $\tau : C_0(X) \longrightarrow \mathbb{C}$ be a tracial state. Then there exists a unique extended Borel measure $\mu : \mathcal{B}(X) \longrightarrow [0, \infty]$ such that μ is finite on compact subsets, $\mu(A) = \sup\{\mu(K), K \subseteq A$ compact$\}$ for any Borel set $A \subseteq X$ and $\tau(f) = \int_X f d\mu$ for any $f \in C_0(X)$.

Definition 8.3.4. Let h be a self-adjoint element of $\mathcal{Z} \otimes \mathcal{K}$. Using the GNS representation, we define $\tau_h := \tau_{|C^*(h)}$, where τ is the unique tracial state on $\mathcal{Z}$. From Riesz-Markov theorem we can consider μ_h, the unique measure obtain from τ_h. Also define $d_\tau(h) := \mu_h(\mathrm{sp}(h) \setminus \{0\})$.

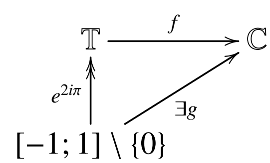
Proposition 8.3.5. *[71, Lemma 6.10], [68, Remark 6.1]*
Let h be a self-adjoint element of $\mathcal{Z} \otimes \mathcal{K}$. Then:
(i) $d_\tau(f(h)) = \mu_h(\mathrm{supp}(f))$ for any $f \in C_0(\mathrm{sp}(h) \setminus \{0\})$.
(ii) $d_\tau(h) = \lim_n \tau(h^{1/n}) = \lim_n \int_{(\mathrm{sp}(h)\setminus\{0\})} t^{1/n} d\mu_h$. In fact, $d_\tau(h)$ is a supremum whenever $h \geq 0$.
(iii) For any $U \subseteq \mathrm{sp}(h) \setminus \{0\}$ open, we have $\mu_h(U) = d_\tau(f(h))$ for any f such that $\mathrm{supp}(f) = U$.

Theorem 8.3.6. *(see e.g [4, §7.3.2])*
Consider $S := \mathbb{N} \sqcup]0, \infty]$ with + coming from the one in $\mathbb{R}_+$. Consider the usual order in both disjoint sets and add a mixed-order as follows: For any $n \in \mathbb{N}$, write $x_n := n \in]0, \infty]$. Then for any $\epsilon > 0$, we have $x_n \leq n \leq x_n + \epsilon$.
Then S is a totally ordered Cu*-semigroup whose compact elements are exactly $\mathbb{N}$. In fact, we have the following* Cu*-isomorphism:*

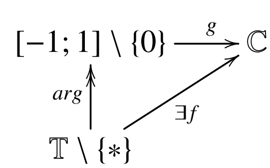
$$\begin{aligned} \mathrm{Cu}(\mathcal{Z}) &\longrightarrow \mathbb{N} \sqcup]0, \infty] \\ [a] &\longmapsto \begin{cases} n \in \mathbb{N}, \text{ if } a \sim_{\mathrm{Cu}} \mathrm{diag}_n(1_{\mathcal{Z}}, ..., 1_{\mathcal{Z}}) \\ d_\tau(a) \in]0, \infty] \text{ else} \end{cases} \end{aligned}$$

Proposition 8.3.7. *Let A be a C^*-algebra and let h be a self-adjoint element of A. Then $C^*(e^{2i\pi h}) \subseteq C^*(h)$. If moreover $\mathrm{sp}(e^{2i\pi h}) \neq \mathbb{T}$, then $C^*(e^{2i\pi h}) = C^*(h)$.*

Proof. First observe that for any (positive) morphism $f : \mathbb{T} \longrightarrow \mathbb{C}$, there exists a (positive) morphism $g : [-1; 1] \setminus \{0\} \longrightarrow \mathbb{C}$ such that the following diagram is commutative:



Conversely, for any (positive) morphism $g :]-1; 1[\setminus \{0\} \longrightarrow \mathbb{C}$, there exists a (positive) morphism $f : \mathbb{T} \setminus \{*\} \longrightarrow \mathbb{C}$ such that the following diagram is commutative:



Now let h be a self-adjoint element of A. Since $\mathrm{sp}(h) \subseteq [-1; 1] \setminus \{0\} \subseteq \mathbb{R}$ and the exponential map is a continuous map from $\mathbb{R}$ to $\mathbb{T}$, by functional calculus we obtain $e^{2i\pi h} \in C^*(h)$. Conversely, since the argument map, inverse of the exponential map, is well-defined from $\mathbb{T} \setminus \{*\}$ to $\mathrm{sp}(h) \subseteq [-1; 1] \setminus \{0\}$, whenever $\mathrm{sp}(e^{2i\pi h}) \subseteq \mathbb{T} \setminus \{*\}$, we obtain that $h \in C^*(e^{2i\pi h})$, which ends the proof. □

Definition 8.3.8. Let $k \in \mathbb{N}$ and let h_k be a positive element of $\mathcal{Z}$ such that $\mathrm{sp}(h_k) = [0, k]$ and $\mu_k = m/k$, where m is the Lebesgue measure on $\mathbb{R}$. We define $u_k := e^{2i\pi h_k}$. Observe that u_k is a unitary element of $\mathcal{Z}$ whose spectrum is $\mathbb{T}$.

Theorem 8.3.9. *For all $k, l \geq 1$, we have that* $\mathrm{Cu}(\varphi_{u_k}) = \mathrm{Cu}(\varphi_{u_l})$. *Moreover, if* $k \equiv l + 1 \mod 2$, *then* $\varphi_{u_k} \nsim_{aue} \varphi_{u_l}$.

Proof. Since $\mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}})$ is generated by $\{1_{|U}\}_{U \in O(\mathbb{T})}$, we only need to check that $\mathrm{Cu}(\varphi_{u_k})(f) = \mathrm{Cu}(\varphi_{u_l})(f)$ for any $f \in C(\mathbb{T})_+$. Let $k \in \mathbb{N}$, let $f \in C(\mathbb{T})_+$ and let u_k be the unitary defined in Definition 8.3.8. We are going to show that $[f(u_k)]$ does not depend on the k chosen and the result will follow.
By Theorem 8.3.6, we know that $[f(u_k)] \in \{0, 1_c\} \sqcup]0, 1_{nc}]$. Moreover, $[f(u_k)] = 0$ if and only if $f(u_k) = 0_{\mathcal{Z}}$. Equivalently, $f = 0_{|\mathbb{T}}$ and $[f(u_k)] = 1_c$ if and only if $f(u_k) = \lambda 1_{\mathcal{Z}}$, that is $f = \lambda 1_{|\mathbb{T}}$.
Now suppose that $[f(u_k)] \in]0, 1_{nc}]$. From the proof of Proposition 8.3.7, we know that there exists $g_k : \mathrm{sp}(h_k) \setminus \{0\} \longrightarrow \mathbb{R}_+$ such that $f(u_k) = g_k(h_k)$ (respectively g_l, h_l and u_l). Moreover, we know that $[f(u_k)]$ is not a compact element and hence $[f(u_k)] = d_\tau(f(u_k))$.
Define $S := \{t \in]0, 1] : f(u_k) \neq 0\}$. We get that $d_\tau(g_k(h_k)) = \mu_{h_k}(\mathrm{supp}\, g_k) = k\mu_{h_k}(S) = m(S)$. We deduce that $d_\tau(g_k(h_k))$ does not depend on the k chosen, which implies that $[f(u_k)]$ does not depend on the k chosen. Thus, we get $\mathrm{Cu}(\varphi_{u_k}) = \mathrm{Cu}(\varphi_{u_l})$.
Now, let us compute $\Delta_\tau(u_k) = [\tau(h_k)]_{\mathcal{Z}_q/\underline{\tau}(\mathrm{K}_0(\mathcal{Z}))}$. In fact, $\Delta_\tau(u_k) = [\tau(h_k)]_{\mathbb{C}/\mathbb{Z}}$ and $\tau(u_k) = \int_0^k (t/k) dm = k/2$. We obtain that $\Delta_\tau(u_k) \neq \Delta_\tau(u_l)$ whenever $k \equiv l + 1 \mod 2$. The conclusion follows using the contraposition of Corollary 8.2.5. □

Proposition 8.3.10. *We claim that the positive element h_k described in Definition 8.3.8 exists in $\mathcal{Z}_+$, for any $k \in \mathbb{N}$.*

Proof. We look for a positive element h in $\mathcal{Z}$ with $\mathrm{sp}(h_k) = [0, k]$ and $\mu_k = m/k$. Equivalently, we look for a *-homomorphism $\varphi_h : C_0(]0, k]) \longrightarrow \mathcal{Z}$.
Consider the following assignment:

$$\begin{aligned} \alpha : \mathrm{Lsc}(]0, k], \overline{\mathbb{N}}) &\longrightarrow \mathbb{N} \bigsqcup]0, \infty] \\ f &\longmapsto \int_0^k (f/k) dm \end{aligned}$$

We integrate step-maps on a compact set. One can then check that α preserves addition, order, and suprema of increasing sequences. Also, by Lemma 6.1.14 we obtain that α preserves $\ll$ and hence α is a Cu-morphism.
By [64, Theorem 2], we know there exists $\varphi : C_0(]0, k]) \longrightarrow \mathcal{Z}$ such that $\mathrm{Cu}(\varphi) = \alpha$ which corresponds to a positive element h. Finally, we have to check that h has the required properties. It is clear that $\mathrm{sp}(h) =]0, k]$. Now let $U \subseteq]0, k]$ be an open set, and let $f \in C_0(]0, k])$

such that $\operatorname{supp}(f) = U$. We know that $d_\tau(f(h)) = \mu_h(U)$. Further, $d_\tau(f(h)) = [f(h)]_{\mathrm{Cu}}$, hence $\mu_h(U) = \int_0^k ([f]/k) d\mathfrak{m} = \mathfrak{m}(U)/k$, which ends the proof. □

Corollary 8.3.11. Cu *does not classify unitary elements of* $\mathcal{Z}$.

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