A new invariant for C^*-algebras

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Plan

- Introduction
- The Cu_1 semigroup
- Recovering existing classification results
- A concrete use of Cu_1
- Classification of unitary elements

Introduction

Historical Background

von Neumann algebras

- Weakly-closed *-subalgebras of bounded operators $\mathcal{B}(H)$.
- Their classification by types was started by Murray and von Neumann in the 30's.
- Nowadays, the classification is complete (notably by Connes).
- Among other tools, projections played a key role.

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Abstract definition

In the 40's, Gelfand and Naimark discovered an abstract way to characterize norm-closed *-subalgebras of bounded operators $\mathcal{B}(H)$: the C^*-algebras.

Introduction

C^*-algebra

Definition

A C^*-algebra A is an algebra over $\mathbb{C}$ with an involution $^* : A \longrightarrow A$, and equipped with an algebra norm $\| \, \|$, such that A is a Banach space and such that A satisfies the C^*-*property*: $\|aa^*\| = \|a\|^2$ for any $a \in A$

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vN algebras, $\mathcal{B}(H)$, $C_0(X)$ - commutative, $M_n(\mathbb{C})$ - finite dimension.

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The classification problem

The aim is to find a functor $\mathcal{F}$ from a subcategory of C^*-algebras to a suitable category $\mathcal{D}$ such that for any two $A, B \in C^*$, if there exists $\alpha : \mathcal{F}(A) \simeq \mathcal{F}(B)$ in $\mathcal{D}$, then there exists a *-isomorphism $\phi : A \simeq B$ and moreover $\mathcal{F}(\phi) = \alpha$.

Introduction

The Elliott classification program

Conjecture - [Elliott, 1989]

$\mathrm{Ell}(A) := ((\mathrm{K}_0(A), \mathrm{K}_0(A)_+, \Gamma(A)), \mathrm{K}_1(A), T(A), r_A)$, where $T(A)$ is the tracial simplex on A and r_A a pairing map between $T(A)$ and $\mathrm{K}_0(A)$, is a complete invariant for simple, separable, nuclear C^*-algebras.

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Result - [Many hands, nowadays]

The classification program is now complete and has provided a successful classification of simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT by the original Elliott invariant.

Introduction

The Cuntz semigroup

Definition - [Cuntz, 1978]

Let A be a C^*-algebra. For any $a, b \in A_+$, we write $a \lesssim_{\mathrm{Cu}} b$, if there exists a sequence $(x_n)_n$ in A such that $a = \lim_{n\in\mathbb{N}} x_n^* b x_n$.

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Definition - [Coward - Elliott - Ivanescu, 2008]

The *Cuntz semigroup* of a C^*-algebra A is $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+/\sim_{\mathrm{Cu}}$.

The category Cu consists of PoM satisfying the following axioms:

(O1): Every increasing sequence has a supremum.

(O2): Any element is the supremum of a $\ll$-increasing sequence, where $x \ll y$ if for any increasing sequence $(z_n)_n$ such that $\sup z_n \geq y$, there exists $z_n \geq x$.

(O3)-(O4): Compatibility axioms.

Introduction

The Cuntz semigroup

Theorem - [Brown - Perera - Toms, 2008]

For simple, unital, stably finite, $\mathcal{Z}$-stable and exact C^*-algebras

$$\mathrm{Cu}(A) \simeq (V(A) \setminus \{0\}) \sqcup L(T(A))$$

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Theorem - [Antoine - Perera - Santiago, 2011]

Let A be C^*-algebra with no K_1-obstructions. Let X be a topological space with $\dim X \leq 1$. *Then* $\mathrm{Cu}(C_0(X) \otimes A) \simeq \mathrm{Lsc}(X, \mathrm{Cu}(A))$.

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Cu *is a complete invariant for direct limits of* NCCW 1 *with* $\mathrm{K}_1 = 0$.

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Remark

Cu captures the lattice of ideals of a separable C^*-algebra.

Introduction

Aim of the thesis

Question

Would it be possible to ‘merge’ the K_1-group together with the Cu-semigroup to obtain a stronger invariant?

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Notation

For the rest of the presentation, we denote by C^* the category of separable C^*-algebras with stable rank one and we focus on this category.

The Cu_1 semigroup

Standard maps

Proposition

Let A be a separable C^*-algebra with stable rank one. Let $a, b \in A_+$. Then $a \precsim_{\mathrm{Cu}} b$ if and only if there exists $x \in A$ such that $a = x^*x$ and $xx^* \in \operatorname{her} b$.

In this case, there exists a partial isometry $\alpha \in A^{**}$ such that

$$\begin{aligned} \theta_{ab,\alpha} : \operatorname{her} a &\hookrightarrow \operatorname{her} b \\ d &\longmapsto \alpha^* d \alpha \end{aligned}$$

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Lemma

In the above context, let $u \in \mathrm{her}\, a^\sim$ be a unitary, then $[\theta^\sim_{ab,\alpha}(u)]_{K_1}$ does not depend on α. We now refer to $\theta^\sim_{ab,\alpha}$ as a standard morphism and we write θ_{ab}.

The Cu_1 semigroup

Definition

Definition

Let $A \in C^*$. Let $a, b \in A_+$ and let u, v be unitary elements of her $a^\sim$ and her $b^\sim$ respectively. We write $(a, u) \lesssim_1 (b, v)$ if:

$$\begin{cases} a \lesssim_{\mathrm{Cu}} b \\ [\theta_{ab}(u)] = [v] \text{ in } \mathrm{K}_1(\mathrm{her}\, b^\sim) \end{cases}$$

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The $\precsim_1$ relation is reflexive and transitive.

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Proposition

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Definition

Let $A \in C^*$. We define

$$\mathrm{Cu}_1(A) := \{(a,u) : a \in (A \otimes \mathcal{K})_+, u \in \mathcal{U}(\mathrm{her}\, a^\sim)\}/\sim_1$$

We equip $\mathrm{Cu}_1(A)$ with addition and order in a usual way.

The Cu_1 semigroup

First properties

Definition

The category $\mathrm{Cu}^\sim$ consists of ordered monoids satisfying the Cuntz axioms and such that $0 \ll 0$.

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Let $A \in C^*$. Then $(\mathrm{Cu}_1(A), \leq)$ is a $\mathrm{Cu}^\sim$-semigroup. In fact, the assignment $A \longmapsto \mathrm{Cu}_1(A)$ is functorial.

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Theorem

The functor $\mathrm{Cu}_1 : C^* \longrightarrow \mathrm{Cu}^\sim$ is continuous. More precisely, given an inductive system $(A_i, \phi_{ij})_{i\in I}$ in C^*, we have:

$$\mathrm{Cu}^\sim - \varinjlim(\mathrm{Cu}_1(A_i), \mathrm{Cu}_1(\phi_{ij})) \simeq \mathrm{Cu}_1(C^* - \varinjlim((A_i, \phi_{ij}))).$$

The Cu_1 semigroup

Positive and compact elements

Definition

Let S be a $\mathrm{Cu}^{\sim}$-semigroup. We define the *positive cone of S* by $S_+ := \{x \in S \mid x \geq 0\}$. We also define a functor $\nu_+ : \mathrm{Cu}^{\sim} \longrightarrow \mathrm{Cu}$.

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Proposition

Let $A \in C^*$ and let $[(a, u)] \in \mathrm{Cu}_1(A)$. Then $[(a, u)]$ is compact if and only if $[a]$ is compact in $\mathrm{Cu}(A)$.

Equivalently, there exists a projection $p \in A \otimes \mathcal{K}$ such that $[p] = [a]$ (Brown - Ciuperca).

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Definition

Let S be a $\mathrm{Cu}^{\sim}$-semigroup. We define $S_c := \{x \in S \mid x \ll x\}$. We also define a functor $\nu_c : \mathrm{Cu}^{\sim} \longrightarrow \mathrm{PoM}^{\sim}$.

The Cu_1 semigroup

Ideals in $\mathrm{Cu}^\sim$

Notion of Ideals and Quotients

- Notion of ideals in the category $\mathrm{Cu}^\sim$ under additional abstract axioms such as being *positively directed* and *positively convex*.
- Positively directed: For any $s \in S$, $\exists p_s \in S$ such that $s + p_s \geq 0$.
- Notion of quotients in the category $\mathrm{Cu}^\sim$.
- Ideals and quotients are positively directed and positively convex $\mathrm{Cu}^\sim$-semigroups.

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- Notion of quotients in the category $\mathrm{Cu}^\sim$.
- Ideals and quotients are positively directed and positively convex $\mathrm{Cu}^\sim$-semigroups.

Theorem

Let $A \in C^*$. The assignment $I \longmapsto \mathrm{Cu}_1(I)$ defines a complete lattice isomorphism from $\mathrm{Lat}(A)$ *to* $\mathrm{Lat}(\mathrm{Cu}_1(A))$.

The Cu_1 semigroup

Link with Cu and K_1

Definition - Proposition

Let S be a positively directed $\mathrm{Cu}^\sim$-semigroup. We define $S_{max} := \{x \in S \mid \text{ if } y \geq x, \text{ then } y = x\}$. S_{max} is an abelian group and we define a functor $\nu_{max} : \mathrm{Cu}^\sim \longrightarrow \mathrm{AbGp}$.

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Theorem

Let $A \in C^*$. Then we have canonical isomorphisms in Cu and AbGp:

$$\mathrm{Cu}_1(A)_+ \simeq \mathrm{Cu}(A) \qquad \mathrm{Cu}_1(A)_{max} \simeq \mathrm{K}_1(A)$$

$$[(a,1)] \longmapsto [a] \qquad [(s_{A\otimes\mathcal{K}}, u)] \longmapsto [u]$$

In fact, we have the natural isomorphisms as follows:

$$\nu_+ \circ \mathrm{Cu}_1 \simeq \mathrm{Cu} \qquad \nu_{max} \circ \mathrm{Cu}_1 \simeq \mathrm{K}_1$$

The Cu_1 semigroup

Quotients and Exactness in $\mathrm{Cu}^\sim$

Theorem

Let $A \in C^*$ and let $I \in \mathrm{Lat}(A)$.

(i) The quotient map $\pi : A \longrightarrow A/I$ induces a $\mathrm{Cu}^\sim$-isomorphism

$$\mathrm{Cu}_1(A)/\,\mathrm{Cu}_1(I) \simeq \mathrm{Cu}_1(A/I).$$

(ii) The following sequence is short exact in $\mathrm{Cu}^\sim$:

$$0 \longrightarrow \mathrm{Cu}_1(I) \xrightarrow{i^*} \mathrm{Cu}_1(A) \xrightarrow{\pi^*} \mathrm{Cu}_1(A/I) \longrightarrow 0\ .$$

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Proposition

Let $S \in \mathrm{Cu}^\sim$ be positively directed. Then, the following sequence is split-exact in $\mathrm{Cu}^\sim$:

$$0 \longrightarrow S_+ \xrightarrow{i} S \underset{q}{\overset{j}{\rightleftarrows}} S_{max} \longrightarrow 0$$

where $j(s) := s + e_{S_{max}}$ and $q(s) := s$.

The Cu_1 semigroup

Exactness in $\mathrm{Cu}^\sim$

Observation

$\mathrm{Cu}^\sim$ is not an abelian category. Thus, S can not be recovered as $S_+ \oplus S_{max}$ a priori.

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Theorem

Let $A, B \in C^*$. Let $\phi : A \longrightarrow B$ be a *-homomorphism. Then the following diagram is commutative with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Cu}(A) & \xrightarrow{i} & \mathrm{Cu}_1(A) & \xrightarrow{j} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Cu}(\phi)} & & \downarrow{\scriptstyle \mathrm{Cu}_1(\phi)} & & \downarrow{\scriptstyle \mathrm{K}_1(\phi)} & & \\
0 & \longrightarrow & \mathrm{Cu}(B) & \xrightarrow{i} & \mathrm{Cu}_1(B) & \xrightarrow{j} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}
$$

The Cu_1 semigroup

Computations

Proposition

If A is simple, then $\mathrm{Cu}_1(A) \simeq (\mathrm{Cu}(A)_* \times \mathrm{K}_1(A)) \sqcup \{0\}$.

If A has no K_1*-obstructions, then* $\mathrm{Cu}_1(A) \simeq \mathrm{Cu}(A)$. *In particular, for any* AF *algebra A,* $\mathrm{Cu}_1(A) \simeq \mathrm{Cu}(A)$.

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Theorem

(i)

$$\mathrm{Cu}_1(C(\mathbb{T})) \simeq \bigsqcup_{U \in \mathcal{O}(\mathbb{T})} \mathrm{Lsc}(U, \overline{\mathbb{N}}_*) \times \bigoplus_1^{n_U} \mathbb{Z}$$
$$\simeq \mathrm{Cu}_1(C(]0,1[)) \sqcup \mathrm{Lsc}(\mathbb{T}, \overline{\mathbb{N}}_*) \times \mathbb{Z}.$$

(ii) $\mathrm{Cu}_1(C(\mathbb{T}))/\,\mathrm{Cu}_1(C_0(]0,1[)) \simeq \overline{\mathbb{N}} \times \{0\}$.

(iii) $\mathrm{Cu}_1(C(\mathbb{T}))_c \simeq (\{n.1_{|\mathbb{T}}\}_{n \in \overline{\mathbb{N}}}) \times \mathbb{Z}$.

The Cu_1 semigroup

Computations

Theorem

Let M_q be a UHF algebra of infinite type. Then:

$$\mathrm{Cu}_1(C(\mathbb{T}) \otimes M_q) \simeq \bigsqcup_{U \in \mathcal{O}(\mathbb{T})} \mathrm{Lsc}(U, \mathbb{N}[\frac{1}{q}]_* \sqcup]0, \infty]) \times (\bigoplus_1^{n_U} \mathbb{Z}[\frac{1}{q}])$$

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Observation

Theoretically, one can use the continuity of Cu_1 to compute the Cu_1-semigroup of $A\mathbb{T}$ algebras. The interval case is computed similarly.

The Cu_1 semigroup

NCCW 1 algebras

Definition

We define a *non-commutative CW complex of dimension 1*, written NCCW 1, as the following pullback in C^*:

$$\begin{array}{ccc} A & \longrightarrow & C([0,1],F) \\ \downarrow & & \downarrow (ev_0, ev_1) \\ E & \xrightarrow[(\phi_0,\phi_1)]{} & F \oplus F \end{array}$$

where E, F are finite dimensional C^*-algebras.

The Cu_1 semigroup

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where E, F are finite dimensional C^*-algebras.

Proposition

There is no $\mathrm{PoM}^{\sim}$*-isomorphism between* $\mathrm{Cu}_1(C(\mathbb{T}))$ *and* $\mathrm{Cu}_1(C([0,1])) \bigoplus_{\mathrm{Cu}_1(\mathbb{C}\oplus\mathbb{C})} \mathrm{Cu}_1(\mathbb{C})$.

Recovering existing classification results

Classification tools

Definition

Let C be a category and let $F : C^* \longrightarrow C$ be a functor. We say that F is a *complete invariant* for a class of C^*-algebras C^*_F, if for any $A, B \in C^*_F$ such that there exists $F(A) \overset{\alpha}{\simeq} F(B)$, then there exists $A \overset{\phi}{\simeq} B$ such that $F(\phi) = \alpha$.

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Definition

Let A, B be C^*-algebras. In the above context, we say that F *classifies homomorphisms* from A to B, if for any $\alpha : F(A) \longrightarrow F(B)$:
(Existence) There exists $\phi : A \longrightarrow B$ such that $F(\phi) = \alpha$.
(Uniqueness) For any other *-homomorphism $\psi : A \longrightarrow B$ such that $F(\psi) = \alpha$, then $\phi \sim_{aue} \psi$.

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The latter notion is stronger than the former.

Recovering existing classification results

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Definition

Let $I : C^* \longrightarrow \mathcal{C}$ and $J : C^* \longrightarrow \mathcal{D}$ be functors. Suppose that there exists $H : \mathcal{D} \longrightarrow \mathcal{C}$ such that $H \circ J \simeq I$. Then we say we can *recover I from J through H*.

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Let $I : C^* \longrightarrow \mathcal{C}$ and $J : C^* \longrightarrow \mathcal{D}$ be functors. Suppose that there exists $H : \mathcal{D} \longrightarrow \mathcal{C}$ such that $H \circ J \simeq I$. Then we say we can *recover I from J through H*.

Theorem

In the above context, if H is faithful then we have the following:

(i) If I is a complete invariant for C^*_I, then so is J.

(ii) If I classifies homomorphisms from C^*_1 to C^*_2, then so does J.

Recovering existing classification results

Classification tools

Remark

We can recover Cu and K_1 from Cu_1 through ν_+ and ν_{max} respectively.

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Corollary

Let $\phi, \psi : A \longrightarrow B$ *be two* **-homomorphisms. If* $Cu_1(\phi) = Cu_1(\psi)$ *then* $Cu(\phi) = Cu(\psi)$ *and* $K_1(\psi) = K_1(\phi)$.

Recovering existing classification results

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Remark

We can recover Cu and K_1 from Cu_1 through ν_+ and ν_{max} respectively.

Corollary

Let $\phi, \psi : A \longrightarrow B$ be two *-homomorphisms. If $\mathrm{Cu}_1(\phi) = \mathrm{Cu}_1(\psi)$ *then* $\mathrm{Cu}(\phi) = \mathrm{Cu}(\psi)$ *and* $\mathrm{K}_1(\psi) = \mathrm{K}_1(\phi)$.

Notations

The category of ordered groups with order-unit will be denoted by AbGp_u and the category of $\mathrm{Cu}^{\sim}$-semigroups with order-unit will be denoted by $\mathrm{Cu}^{\sim}_u$.

Recovering existing classification results

The K_* group

Definition - [Elliott, 1993]

Let A be a (unital) C^*-algebra. We define $K_*(A) := K_0(A) \oplus K_1(A)$. We also define $K_*(A)_+ := \{([p]_{K_0(A)}, [v]_{K_1(A)})\}$, where v is a unitary in the corner $p(A \otimes \mathcal{K})p$. Finally, we define $1_{K_*(A)} := ([1_A]_{K_0}, [1_A]_{K_1})$.

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Theorem - [Elliott - Gong, 1996]

From the above, we obtain a functor $K_* : AH_d \longrightarrow \mathrm{AbGp}_u$ *such that:*

(i) The functor K_* *is a complete invariant for (unital)* AH_d *algebras of real rank zero.*

(ii) The functor K_* *classifies homomorphisms of* $A\mathbb{T}$ *algebras of real rank zero.*

Recovering existing classification results

Recovering K_* from Cu_1

Theorem

The functor $H : \mathrm{Cu}_u^\sim \longrightarrow \mathrm{AbGp}_u$ *that sends* $(S,u) \in \mathrm{Cu}_u^\sim$ *to* $(\mathrm{Gr}(S_c), S_c, u) \in \mathrm{AbGp}_u$ *yields a natural isomorphism* $H \circ \mathrm{Cu}_{1,u} \simeq \mathrm{K}_*$. *Moreover, if we restrict the domain of* H *to* $\mathrm{Cu}_{u,alg}^\sim$, *then* H *becomes a faithful functor.*

Recovering existing classification results

Recovering K_* from Cu_1

Theorem

The functor $H : Cu_u^\sim \longrightarrow AbGp_u$ *that sends* $(S, u) \in Cu_u^\sim$ *to* $(Gr(S_c), S_c, u) \in AbGp_u$ *yields a natural isomorphism* $H \circ Cu_{1,u} \simeq K_*$. *Moreover, if we restrict the domain of* H *to* $Cu_{u,alg}^\sim$, *then* H *becomes a faithful functor.*

Corollary

By restricting to the category $Cu_{u,alg}^\sim$, *we can fully recover* K_* *from* $Cu_{1,u}$ *through* H. *A fortiori, we have:*

(i) $Cu_{1,u}$ *is a complete invariant for* AH_d *algebras of real rank zero.*

(ii) $Cu_{1,u}$ *classifies homomorphisms of* $A\mathbb{T}$ *algebras with real rank zero.*

A concrete use of Cu_1

Preliminaries

Aim

We construct two C^*-algebras whose K-Theory and Cu-semigroup are isomorphic, nevertheless $\mathrm{Cu}_1(A) \not\simeq \mathrm{Cu}_1(B)$ and thus $A \not\simeq B$.

A concrete use of Cu_1

Preliminaries

Aim

We construct two C^*-algebras whose K-Theory and Cu-semigroup are isomorphic, nevertheless $\mathrm{Cu}_1(A) \not\simeq \mathrm{Cu}_1(B)$ and thus $A \not\simeq B$.

Observation

The construction involves inductive limits of NCCW_1-algebras and hence it deals with inductive limits of lower semicontinuous functions.

A concrete use of Cu_1

Piecewise characteristic functions

Definition

Let X be either the interval or the circle. Let $g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$.
We say that g is an 3-*piecewise characteristic function of size n* if $g \ll \infty$ and if there exist $s_1, ..., s_{3^n} \in \mathbb{N}$ such that $g_{|U_k} = s_k$ for any $1 \leq k \leq 3^n$, where $\{\overline{U_k}\}_{k=1}^{3^n}$ is the canonical 1-thin cover of X of size $1/3^n$.
The set of 3-piecewise characteristic functions of size n of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ will be denoted by $\chi_n(X)$.

A concrete use of Cu_1

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The set of 3-piecewise characteristic functions of size n of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ will be denoted by $\chi_n(X)$.

Observation

Whenever $n < m$, we have $\chi_n(X) \subset \chi_m(X)$. Furthermore, χ_n and $\bigcup\limits_{n\in\mathbb{N}} \chi_n$ are PoM.

A concrete use of Cu_1

Generators and basis

Proposition - [Canonical decomposition - Generators]

In the above context, any $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ *can be uniquely described by a* $\subseteq$*-decreasing sequence of open sets in* X*. We say that* $\{1_U\}_{U \in \mathcal{O}(X)}$ *generates* $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

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Proposition - [Countable basis - Density]

$\bigcup_{n\in\mathbb{N}} \chi_n(X)$ is a countable basis of $\mathrm{Lsc}(X, \overline{\mathbb{N}})$. That is, for any $f \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$, there exists a $\ll$-increasing sequence $(g_l)_l$ in $\bigcup_{n\in\mathbb{N}} \chi_n(X)$ such that $\sup_{l\in\mathbb{N}} g_l = f$.

We say that $\bigcup_{n\in\mathbb{N}} \chi_n(X)$ is dense in $\mathrm{Lsc}(X, \overline{\mathbb{N}})$.

A concrete use of Cu_1

Cu-metrics

Definition

Let $\alpha, \beta : S \longrightarrow T$ be Cu-morphisms and let Γ be a finite subset of S. We say that α *is* $\approx$*-equivalent to* β *on* Γ, and we write $\alpha \underset{\Gamma}{\approx} \beta$, if for any $g' \ll g$ in Γ, $\alpha(g') \ll \beta(g)$ and $\beta(g') \ll \alpha(g)$.

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Definition - [Cu-metrics]

Let $\alpha, \beta : \mathrm{Lsc}(X, \overline{\mathbb{N}}) \longrightarrow T$ be Cu-morphisms. We define

The Cu-metric, denoted $d_{\mathrm{Cu}}(\alpha, \beta)$ as:

$$\inf\{r > 0 \mid \forall U \in \mathcal{O}(X), \alpha(1_U) \leq \beta(1_{U_r}) \text{ and } \beta(1_U) \leq \alpha(1_{U_r})\}.$$

The Cu-semimetric, denoted $dd_{\mathrm{Cu}}(\alpha, \beta)$ as:

$$\inf\{1/3^n \mid \forall g' \ll g \in \Gamma_n(X), \alpha(g') \leq \beta(g) \text{ and } \beta(g') \leq \alpha(g)\}.$$

A concrete use of Cu_1

Cu-metrics

Proposition

If $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$, then $\alpha \underset{\Gamma_m}{\approx} \beta$ for any $m < n$. Conversely, if $\alpha \underset{\Gamma_n}{\approx} \beta$, then $dd_{\mathrm{Cu}}(\alpha,\beta) \leq 1/3^n$.

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Corollary

In the above context, the following are equivalent:

(i) $d_{\mathrm{Cu}}(\alpha,\beta) = 0$.

(ii) $dd_{\mathrm{Cu}}(\alpha,\beta) = 0$.

(iii) $\alpha \underset{\Gamma_n}{\approx} \beta$, for any $n \in \mathbb{N}$.

(iv) $\alpha = \beta$.

A concrete use of Cu_1

Intertwining theorems

Theorem - [Intertwining Theorem]

Consider two inductive sequences as follows:

$$... \rightrightarrows \mathrm{Lsc}(X, \overline{\mathbb{N}}) \underset{\tau_{ii+1}}{\overset{\sigma_{ii+1}}{\rightrightarrows}} \mathrm{Lsc}(X, \overline{\mathbb{N}}) \rightrightarrows ...$$

Let S and T be their respective inductive limits. Suppose there exists a strictly increasing sequence $(n_i)_i$ in $\mathbb{N}$ such that:

(i) $\sigma_{ii+1} \underset{\Gamma_{n_i}}{\approx} \tau_{ii+1}$, for any $i \in \mathbb{N}$.

(ii) For any $i \leq j$ and any $l \in \mathbb{N}$, we have $\sigma_{ij}(\chi_l), \tau_{ij}(\chi_l) \subseteq \chi_l$.

Then, $S \simeq T$ as Cu*-semigroups.*

A concrete use of Cu_1

Intertwining theorems

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Then, $S \simeq T$ as Cu*-semigroups.*

Observation

We have been able to extend the intertwining theorem to sequences of finite direct sums of $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ and $\mathrm{Cu}(\mathcal{I}_q^n)$.

A concrete use of the invariant

Evans-Kishimoto folding interval algebras

Definition - [Evans - Kishimoto, 1991]

Let $q \in \mathbb{N}$ and let e and $f := (1_{M_q} - e)$ be two (non-trivial) projections in M_q. For any $n \in \mathbb{N}$, we define $\mathcal{I}^n_{q,e}$ as the following pullback:

$$\begin{array}{ccc}
\mathcal{I}^n_{q,e} & \xrightarrow{\pi_1} & C([0,1], \bigotimes_1^n M_q) \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle (ev_0, ev_1)} \\
(\bigotimes_1^{n-1} M_q) \oplus (\bigotimes_1^{n-1} M_q) & \xrightarrow[(i_0^n, i_1^n)]{} & (\bigotimes_1^n M_q) \oplus (\bigotimes_1^n M_q)
\end{array}$$

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Proposition

$K_0(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z}$ $\qquad$ $Cu(\mathcal{I}^n_{q,e}) \simeq \{f \in Lsc([0,1], \frac{1}{q^n}\overline{\mathbb{N}}) \mid f(0), f(1) \in \frac{q}{q^n}\overline{\mathbb{N}}\}$

$K_1(\mathcal{I}^n_{q,e}) \simeq \mathbb{Z}/q\mathbb{Z}$

A concrete use of the invariant

Construction of A and B

Proposition - [Evans - Kishimoto, 1991]

In the above context, we have the following *-homomorphism for any $n \in \mathbb{N}$:

$$\begin{aligned} \psi_{n,e} : \mathcal{I}^{n}_{q,e} &\longrightarrow \mathcal{I}^{n+1}_{q,e} \\ f &\longmapsto f(\xi_0) \otimes e + f(\xi_1) \otimes f \end{aligned}$$

where $\xi_0 : t \longmapsto t/2$ *and* $\xi_1 : t \longmapsto 1 - t/2$. *We also have:*

$$\mathrm{K}_0(\psi_{n,e}) : \mathbb{Z} \xrightarrow{\times q} \mathbb{Z} \qquad \mathrm{K}_1(\psi_{n,e}) : \mathbb{Z}/q\mathbb{Z} \xrightarrow{\times \mathrm{rank}(e)} \mathbb{Z}/q\mathbb{Z}.$$

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Remark

The key point of the construction lies in choosing matrices of size $q_i := p_{i-1}p_i$ and taking projections of respective ranks p_{i-1}, p_i for A, B where $(p_i)_i$ is the sequence of prime numbers.

A concrete use of the invariant

Construction of A and B

Definition - [Construction of the blocks]

$$
\begin{aligned}
A_0 &= C[0,1] \\
A_1 &= M_{q_0^{m_0}}(\mathcal{I}^1_{q_0}) \oplus C[0,1] \\
A_2 &= M_{q_0^{m_0} q_0^{m_1}}(\mathcal{I}^2_{q_0}) \oplus M_{q_1^{m_1}}(\mathcal{I}^1_{q_1}) \oplus C[0,1] \\
&\vdots \\
A_n &= M_{q_0^{m_0} \dots q_0^{m_{n-1}}}(\mathcal{I}^n_{q_0}) \oplus \dots \oplus M_{q_i^{m_i} \dots q_i^{m_{n-1}}}(\mathcal{I}^{n-i}_{q_i}) \oplus \dots \oplus C[0,1] \\
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&\vdots
\end{aligned}
$$

Observation

Observe that for each q_i, there is a projection e^i_A of M_{q_i} such that $\operatorname{rank}(e^i_A) = p_{i-1}$ that we have omitted in the notation.

A concrete use of the invariant

Construction of A and B

Definition - [Construction of the morphisms]

For any $0 \leq i \leq n-1$ we define:

$$\phi^{i}_{nn+1} : M_{q_i^{m_i}\dots q_i^{m_{n-1}}}(\mathcal{I}^{n-i}_{q_i}) \longrightarrow M_{q_i^{m_i}\dots q_i^{m_n}}(\mathcal{I}^{n-i+1}_{q_i})$$

$$f \longmapsto \begin{pmatrix} f(\xi_0)\otimes e^i_A + f(\xi_1)\otimes f^i_A & & & \\ & f(1/r)\otimes 1_{M_{q_i}} & & \\ & & \dots & \\ & & & f(r-1/r)\otimes 1_{M_{q_i}} \end{pmatrix}$$

A concrete use of the invariant

Construction of A and B

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Definition - [Construction of the algebras]

$A := \varinjlim(A_n, \phi_{nm})$. We construct B similarly by taking $\mathrm{rank}(e^i_B) = p_i$.

A concrete use of the invariant

Construction of A and B

Definition - [Construction of the morphisms]

For any $0 \leq i \leq n-1$ we define:

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Definition - [Construction of the algebras]

$A := \varinjlim(A_n, \phi_{nm})$. We construct B similarly by taking $\mathrm{rank}(e^i_B) = p_i$.

Proposition

Both A and B are separable unital C^*-algebras of stable rank one.

A concrete use of the invariant

Computation of the K-Theory

Lemma

(i) Any simple ideal of A is of the form $\mathfrak{i}_n := \varinjlim_{m>n}(I_{m,n}, \phi_{mm'}{}_{|I_{m,n}})$ *for some* $n \in \mathbb{N}$.

(ii)
$$A/\bigoplus_{n\in\mathbb{N}} \mathfrak{i}_n \simeq B/\bigoplus_{n\in\mathbb{N}} \mathfrak{j}_n \simeq \mathbb{C}.$$

A concrete use of the invariant

Computation of the K-Theory

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(ii)

$$A/\bigoplus_{n\in\mathbb{N}} \mathfrak{i}_n \simeq B/\bigoplus_{n\in\mathbb{N}} \mathfrak{j}_n \simeq \mathbb{C}.$$

Theorem

Using the 6-term exact sequence and the above, we get:

$$\mathrm{K}_0(A) \simeq \mathrm{K}_0(B).$$

$$\mathrm{K}_1(A) \simeq \mathrm{K}_1(B).$$

A concrete use of the invariant

The Cuntz semigroup of A and B

Lemma

For any $n \in \mathbb{N}$, $\mathrm{Cu}(A_n) \simeq \mathrm{Cu}(B_n)$, *that we write* S_n. *Now consider* $\alpha_{nn+1} := \mathrm{Cu}(\phi_{nn+1})$ *and* $\beta_{nn+1} := \mathrm{Cu}(\psi_{nn+1})$. *We have*

$\alpha_{nn+1}, \beta_{nn+1} : S_n \longrightarrow S_{n+1}$.

Then $dd_{\mathrm{Cu}}(\alpha_{nn+1}, \beta_{nn+1}) \leq 1/3^n$.

A concrete use of the invariant

The Cuntz semigroup of A and B

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Then $dd_{\mathrm{Cu}}(\alpha_{nn+1}, \beta_{nn+1}) \leq 1/3^n$.

Theorem

The approximate intertwining theorem gives us $\mathrm{Cu}(A) \simeq \mathrm{Cu}(B)$.

A concrete use of the invariant

The Cuntz semigroup of A and B

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Then $dd_{\mathrm{Cu}}(\alpha_{nn+1}, \beta_{nn+1}) \leq 1/3^n$.

Theorem

The approximate intertwining theorem gives us $\mathrm{Cu}(A) \simeq \mathrm{Cu}(B)$.

Theorem

There is no $\mathrm{Cu}^{\sim}$-isomorphism between $\mathrm{Cu}_1(A)$ and $\mathrm{Cu}_1(B)$. A fortiori, $A \not\simeq B$.

A concrete use of the invariant

Sketch of the proof

Suppose $\alpha : \mathrm{Cu}_1(A) \longrightarrow \mathrm{Cu}_1(B)$ is a $\mathrm{Cu}^\sim$-isomorphism. For any $n \in \mathbb{N}$ there exists a unique $m \in \mathbb{N}$ such that $\alpha_{|\,\mathrm{Cu}_1(\mathfrak{i}_n)} : \mathrm{Cu}_1(\mathfrak{i}_n) \simeq \mathrm{Cu}_1(\mathfrak{j}_m)$.

In fact, we have the following row-exact and commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Cu}(\mathfrak{i}_n) & \xrightarrow{i} & \mathrm{Cu}_1(\mathfrak{i}_n) & \xrightarrow{j} & \mathrm{K}_1(\mathfrak{i}_n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{0|\,\mathrm{Cu}(\mathfrak{i}_n)}} & & \downarrow{\scriptstyle \alpha_{|\,\mathrm{Cu}_1(\mathfrak{i}_n)}} & & \downarrow{\scriptstyle \alpha_{\mathfrak{i}_n}} & & \\
0 & \longrightarrow & \mathrm{Cu}(\mathfrak{j}_m) & \xrightarrow[i]{} & \mathrm{Cu}_1(\mathfrak{j}_m) & \xrightarrow[j]{} & \mathrm{K}_1(\mathfrak{j}_m) & \longrightarrow & 0
\end{array}$$

We deduce that $\mathrm{K}_0(\mathfrak{i}_n) \simeq \mathrm{K}_0(\mathfrak{j}_m)$ and $\mathrm{K}_1(\mathfrak{i}_n) \simeq \mathrm{K}_1(\mathfrak{j}_m)$. However:

$$\begin{cases} \mathrm{K}_0(\mathfrak{i}_n) \simeq \mathrm{K}_0(\mathfrak{j}_m) \text{ if and only if } n = m \\ \mathrm{K}_1(\mathfrak{i}_n) \simeq \mathrm{K}_1(\mathfrak{j}_m) \text{ if and only if } n + 1 = m \end{cases}$$

Classification of unitary elements

Observation

Let A be a unital C^*-algebra. Then

$$\varphi : \mathcal{U}(A) \simeq \mathrm{Hom}_{C^*,1}(C(\mathbb{T}), A)$$
$$u \longmapsto \varphi_u$$

where $\varphi_u(f) := f(u)$ for any $f \in C(\mathbb{T})$.

Classification of unitary elements

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where $\varphi_u(f) := f(u)$ for any $f \in C(\mathbb{T})$.

Theorem

Let $A := \varinjlim_n (A_n, \phi_{nm})$ *be a (unital)* AF *algebra. Let* u, v *be unitary elements of* A *such that* $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$*. Then* $u \sim_{aue} v$.

Classification of unitary elements

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Let A be a unital C^*-algebra. Then

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Theorem

Let $A := \lim\limits_{\longrightarrow n}(A_n, \phi_{nm})$ be a (unital) AF *algebra. Let u, v be unitary elements of A such that* $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$. *Then $u \sim_{aue} v$.*

Need of K_1 information

- An example in $C[0,1] \otimes M_{2^\infty}$
- An example in the Jiang-Su algebra

An example in $C[0,1] \otimes M_{2^\infty}$

Construction of the unitary elements

We construct

$$u : [0,1] \longrightarrow M_{2^\infty} \qquad v : [0,1] \longrightarrow M_{2^\infty}$$
$$t \longmapsto w \qquad t \longmapsto e^{2i\pi t}.w$$

where $w := \lim_{n\in\mathbb{N}} (\mathrm{diag}(e^{2ik\pi/2^n})_{k=0}^{2^n-1})$.

Theorem

We have $\mathrm{Cu}(\varphi_u) = \mathrm{Cu}(\varphi_v)$, however u is not approximately unitarily equivalent to v.

Thank you for your attention